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ON THE RELATION BETWEEN RAINFALL AND STREAM FLOW—II

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PART I

This article, the second of a series on the subject,[1] presupposes an acquaintance with the first article. Here, instead of considering the drainage area to be rectangular, the theory is extended to irregularly-shaped drainage areas such as are actually met with on the earth's surface.

Throughout this article, as in the first paper, the rate of rainfall, the dryness of the soil, and the velocity of the water are considered to be constant. Moreover, evaporation is neglected as before. However, in the last section of this paper it is pointed out that the methods here employed for irregularly-shaped drainage areas are entirely general and can be readily used when the aforementioned restrictions are removed.

In the first paper no mention was made of the tributaries that streams which exceed the size of brooks invariably have; neither was anything said about the distance which water has to flow overland in order to reach stream channels. Furthermore, while the first paper discussed the discharge from a rectangular drainage area, no precise explanation was given as to what was meant by a rectangular drainage area. Does a rectangular drainage area imply one whose divide outlines a rectangle regardless of how the river and its tributaries meander within the rectangle? These questions were purposely ignored for several reasons: The first paper was essentially introductory; the conditions there treated were very much idealized; it was believed that the discussion of the questions just mentioned belonged in this second article. It is necessary that these neglected features be made clear before irregularly shaped drainage areas can be adequately treated, and therefore they will now be discussed. In discussing them it will be simplest to explain certain steps essential to the application of this theory, which explanations at first sight might appear more properly to belong in the second group of articles.

Despite the features neglected in the first paper, it will have been observed that the distance which water has to travel in order to reach the gage plays an important part in determining the discharge from rainfall. Clearly, in the case of a specific drainage area there may be a large subarea from which water has to travel an approximate distance of a certain number of miles, while for an approximate distance of a different number of miles the corresponding subarea may be much less. In other words, for any given interval of distance which water has to travel, the subareas may be quite different; for example, the subarea from which water has to travel from 10 to 20 miles to reach the gage may be much less than the subarea from which water has to travel from 40 to 50 miles. In order to develop a general theory it is necessary to take this fact into account. Such is the main purpose of this second paper, and it is here shown that, regardless of how irregular any natural drainage area may be, systematic methods can be used for predicting its discharge from rainfall.

In order to apply the theory which will be developed in part II it is necessary to have accurate maps of the river basin whose discharge it is desired to predict. Maps like the topographic maps made and issued by the United States Geological Survey will answer the purpose very nicely.

If the proper maps are available, then the first step in applying the theory is to outline the drainage area on the maps. This is accomplished by drawing a line which begins at the gaging station, follows the divide, and finally ends at the gaging station, thus inclosing the drainage area. After this it is necessary to determine the greatest distance which water has to travel in order to reach the gaging station. This distance should then be divided into a convenient number of equal parts. Say the greatest distance which water has to travel is 300 miles. If this distance of 300 miles be divided into 12 equal parts, each part will be of length 25 miles. Steps should now be taken to outline the several subareas from which water, in order to reach the gaging station, has to travel more than 0 miles and less than 25 miles, more than 25 miles and less than 50 miles, more than 50 miles and less than 75 miles, and so on. To carry this out it is necessary to measure, with a suitable measuring device such as a chartometer, up the main stream and up each tributary, marking off points corresponding to 25 miles, 50 miles, 75 miles, and so on, always being careful to follow the thread of the stream while measuring. After these points have been located, all of them which are 25 miles from the gage should be connected by a smooth line (curve) beginning at the divide[2] on one side of the main stream and ending at the divide on the opposite side. All points which are 50 miles from the gage should be similarly connected, and so on. Clearly each tributary may have tributaries of its own and these tributaries may also have tributaries and so on to mere brooks or ditches. Hence, the process of finding the boundaries of the several aforementioned subareas is a slow and tedious task, but it is perfectly straightforward and can be carried out easily (though slowly) if the proper maps are available.

From the above explanation it will be seen that the lines (curves) which outline the subareas are lines such that from all points on any one of them water has to travel equal distances in order to reach the gage. For this reason they will, in the future, be called *equal water travel* lines.

Where equal water travel lines cross stream channels they do so at right angles. Thus, in a certain sense, the

[1] The first paper appears in MONTHLY WEATHER REVIEW, 62: 315–322, September 1934.

[2] The term "divide" will be used in this series of articles for the line representing the ridge of high ground inclosing the drainage area that contributes to the discharge at the gaging station. The four terms "drainage area", "drainage basin", "river basin", and "watershed" will be used synonymously for the area inclosed by the divide. The term "water parting" will be used for any line representing a ridge of the high ground *within* the river basin which separates the areas drained by two tributaries.

net of stream channels and the equal water travel lines form a mutually orthogonal system. In this respect equal water travel lines are like contour lines, for the latter also cross stream channels at right angles. However, equal water travel lines should not be confused with contour lines, as the two have no other property in common.

After the equal water travel lines have been drawn on the maps as explained above, the subareas comprised between successive equal water travel lines should be measured. These subareas can be quickly and easily obtained by means of a planimeter.

If these subareas be plotted on a diagram, on which the midpoint distances of the subareas above the gaging station are abscissas (in the above example these midpoint distances are 12½, 37½, 62½ miles, and so on, respectively) and the corresponding subareas divided by the distance between consecutive equal water travel lines are ordinates, a histogram of the drainage area is obtained. This histogram shows the true shape of the drainage area.

The channel which is followed by the water that flows the greatest distance in reaching the gage will be considered as the main stream. On the above-mentioned diagram the main stream, *rectified*, forms the axis of abscissas.

Clearly in the above example it was not necessary to divide the 300 miles into 12 equal parts. The 300 miles could have been divided into a greater number of equal parts, thus making the distance between consecutive equal water travel lines smaller. Obviously, a very accurate histogram can be constructed by making the distance between consecutive equal water travel lines very small. After the histogram has been completed, a mathematical curve can be fitted to it, the equation of which will give the width of the drainage area at any distance above the gaging station.

In this paper no histogram has been constructed for a specific river basin; the above explanation has been given to show that the theory developed in this second paper is very general and can be applied to any river basin for which maps are available. In the second group of papers the complete process of making one or more histograms for one or more actual river basins will be given.

The distance which water has to travel from where it falls as rain over a drainage area, to a gaging station on the stream, can be divided into two parts, viz: (1) that part where the water travels in the stream channel (or the stream channel with its adjacent flood plain), and (2) that part where the water travels over the land before reaching a permanent stream channel such as a gully or brook. The question of overland travel of water was tacitly ignored in the first paper; this was done in order to keep the mathematical developments there (the first paper being essentially introductory in character) as simple as possible. The first section of part II discusses the question of overland travel. It is there shown that the error involved in neglecting overland travel entirely, as was done in the first paper, is small. A helpful corollary can be drawn from this conclusion: In drawing lines of equal water travel on maps it is not necessary to use extreme care in drawing an equal water travel line *overland* from one tributary to the adjacent tributary. As the location of the points on the tributaries where equal water travel lines cross them is a slow process, the fact that the lines can be drawn from tributary to tributary by personal judgment rather than by exact measurement adds to the ease with which the theory can be applied. Of course, when the points where equal water travel lines cross the several tributaries are being located, great care should be exercised.

The question of the number of parts into which the greatest distance that water has to travel in order to reach the gage (the 300 miles in the illustration above) should be divided, i. e., how many equal water travel lines should be drawn when dealing with a given watershed, in order to obtain reasonably accurate results, will be taken up in the second group of articles on this subject.

If the greatest distance which water has to travel in order to reach the gage be divided into a very large number of parts, the histogram obtained by the above described procedure will approach a smooth curve; or, putting it differently, if the distance between consecutive equal water travel lines be made infinitely small, then the histogram approaches a smooth curve as a limit. This curve will be termed the *drainage area curve*. Its abscissas are distances above the gaging station, and its ordinates are the widths of the drainage area. These widths must be obtained by the processes above outlined. Thus, in general, the width of the drainage area at an arbitrary distance above the gage will not be the length of an equal water travel line corresponding to this distance; neither will it be the width of the watershed as a crow flies.

Obviously, it is impracticable in constructing a histogram of a drainage area to make the distance between consecutive equal water travel lines very small. Hence the actual mathematical curve which is fitted to the histogram of the drainage area will not coincide with the theoretical drainage area curve. However, it should approximately follow the theoretical drainage area curve. This point will be further discussed in the second group of articles; in the present paper we assume that this problem will cause no difficulty, and the two terms *drainage area curve* and *histogram of the drainage area* are used more or less synonymously in the first group of articles.

Summing up the above explanations, we can say that the shape of the divide has no bearing whatever on what is meant by the shape of the drainage area. The true shape of the drainage area is obtained by an involved process. In particular we say a drainage area is rectangular when the areas comprised between consecutive equal water travel lines are all equal to each other. Since the main stream is the axis of abscissas, we can also say a drainage area is rectangular when its drainage area curve is a straight line parallel to the axis of abscissas. In general, by the shape of a drainage area is meant the shape of the area enclosed by the drainage area curve and the axis of abscissas.

Two new terms are introduced in this paper; they will now be defined. The first derivative of the discharge with respect to time is here defined as the *rate of discharge*. The *rate of discharge* at any given time is the change in discharge per unit time at that time, and is measured in mile inches per hour per hour. The second derivative of the discharge with respect to time is here defined as the *discharge tendency*. The *discharge tendency* at any given time is the change in the rate of discharge per unit time at that time, and is measured in mile inches per hour per hour per hour.

As pointed out in the first paper, both the volume of discharge and the discharge are functions of the time. Likewise, the rate of discharge and the discharge tendency are functions of the time. If any one of the four quantities, volume of discharge, discharge, rate of discharge, or discharge tendency, be plotted as ordinates on a graph with time as abscissas we obtain a curve. The nomenclature used for these curves is in accordance with the variable that forms the ordinate. To illustrate, figure 4 is a volume-of-discharge *curve;* figure 5 is a discharge *curve*, and so on.

Having considered the questions of overland travel, tributaries, and the precise meaning to be attached to the term "shape of the drainage area", and having defined the new terms introduced in this paper, it now remains, in order to carry out the plan of describing in the first part of each paper what is done mathematically in the second part, to explain the remaining sections of part II:

In section 2 of part II a drainage area whose histogram has the shape of a triangle is discussed. It is there pointed out that even so simple a drainage area leads to equations which cannot be explicitly solved in literal form. It is also pointed out that in this special case, for short rains, that is to say, rains whose duration is less than the time required for water to flow from the most distant part of the drainage area to the gage (compare the comments made in connection with equation (9) in the first paper), a different equation must be solved to obtain the maximum discharge from that which is necessary when the rains are long.

The third section takes up the special case of a drainage area whose histogram has the shape of an ellipse. The discussion there shows that in selecting a curve by which the histogram is to be represented, great care must be taken lest expressions which cannot be integrated be encountered. It is also shown that even though we may not always be able to solve explicitly for the time of the crest, we can, nevertheless, always express the maximum discharge explicitly in literal form. This is a very important point, and will be of the utmost assistance in applying the theory.

Clearly there will be few drainage areas whose histograms can be reasonably well represented by simple geometrical curves such as a rectangle, triangle, or ellipse. For this reason the fourth section takes up the general case where the drainage area curve is expressed by any function whatever.

In this fourth section equations analogous to the equations of the first paper are derived. In addition to enabling us to predict the maximum discharge and its time of occurrence, these equations reproduce the complete hydrograph which results from a single rain. It is there pointed out that for some drainage areas a single rain can cause only one crest, while in other drainage areas it may cause more than one crest, and the precise conditions which the drainage area curve must satisfy in order that a single rain may cause a single crest are given. However, it is further pointed out that if the rain lasts long enough for water to flow from the farthest part of the watershed to the gage, then a single rain can cause only one crest, regardless of the shape of the drainage area.

It is also shown in section 4 that the maximum discharge from a watershed which results from a given rain is equal to the discharge from that portion of the watershed which lies between two equal water travel lines in a steady state. The expression "steady state" here means that the rain has lasted so long that there is just as much water flowing off the soil as there is rain falling upon it; one of the equal water travel lines is that which lies at the greatest distance which water has to travel to reach the gage, the other lies at that distance above the gage which water would travel in the interval beginning with the end of the rain and ending with the time of the crest. However, for infinitely short rains, it is shown that the maximum discharge has a still simpler interpretation, and is equal to the product of the depth of the rain times the velocity of the water times the width of the drainage area at the equal water travel line corresponding to the time of the crest.

Finally, section 4 shows that both the discharge and rate of discharge curves are everywhere continuous, except that for infinitely short rains the rate of discharge curve may have a discontinuity at the time of the crest; and that the discharge tendency curve is continuous except at certain points.

The fifth section discusses a drainage area curve which overcomes all the difficulties raised in the second and third sections. The purpose of the sixth and last section was mentioned in the second paragraph above.

PART II

FOREWORD

The figures in this series of articles are numbered consecutively for the series as a whole, and not separately for each article.

Two methods are used in numbering the equations. When an equation of any preceding paper is generalized in a subsequent paper, then the number of the equation as it first appears in the series will be preceded by a letter. Thus in this paper all of the equations in section 2 of the first article are generalized, hence these generalized equations are preceded by the letter B. The letter C will be similarly used in the third article, and so on. Should it be necessary to give the same equation two separate generalizations in the same subsequent article then the letter will be primed.

Those equations which are not generalizations of equations previously given will be numbered consecutively for the entire series. It is hoped that this dual system of numbering the equations will cause no confusion.

The mathematics used in this article is but slightly more advanced than that in the first one; and every effort has been made, consistent with keeping the cost of publication reasonable, to have all developments clear.

SECTION 1: OVERLAND TRAVEL

In order to facilitate the mathematical treatment of overland travel, it is here assumed that the water travels overland in a direction perpendicular to the stream channel. Naturally this is not always the case, but it will be clear to the reader that this assumption is permissible since the complete neglect of overland travel involves only a small error.

Consider a rectangle whose width is W and whose length is L. Take the X-axis as the line parallel to and midway between the lines forming the sides (length) of the rectangle. Take the origin at one end of the rectangle and assume that the gaging station is at the origin and that the stream channel coincides with the X-axis. Let w be the distance from the stream channel to the infinitesimal *area* $dwdx$; the definitions of all other symbols were given in the first paper.

Now at the time t each infinitesimal area, $dwdx$, contributes to the discharge at the gaging station, y, not its discharge at the time t but its discharge at the time t diminished by the time required for the water to flow from $dwdx$ to the gaging station. Clearly the time required for the water to flow from $dwdx$ to the gaging station is the time required for the water to flow from $dwdx$ to the stream channel plus the time required for the water to flow from where the discharge from $dwdx$ reaches the stream channel to the gaging station. Since the velocity of the water is v, these times are w/v and x/v, respectively; and therefore the time required for the water to flow from $dwdx$ to the origin is $(w+x)/v$. The discharge, y, is the sum of all the discharges from the infinitesimal areas $dwdx$; it is therefore (under properly chosen conditions) equal to the double integral (see fig. 2):

$$\int\int r\left\{1-e^{-\frac{1}{c}\left(t-\frac{x+w}{v}\right)}\right\}dwdx.$$

If the time t be taken sufficiently large so that the water from the area $dwdx$ farthest from the gaging sta-

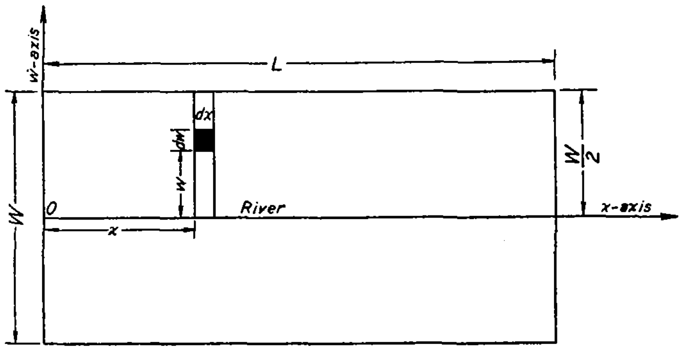


FIGURE 2.

tion has had time to reach the gage (in other words if $tv \geqq L+\frac{1}{2}W$), then the limits on the above integrals are $\frac{1}{2}W$ and $-\frac{1}{2}W$, and L and 0; and on performing the indicated integration and substituting these limits we obtain the equation:

$$y=r\left[WL-2c^2v^2e^{-\frac{t}{c}}\left(e^{\frac{W}{2cv}}-1\right)\left(e^{\frac{L}{cv}}-1\right)\right]. \qquad (10)$$

Equation (10) holds on the range $\frac{L+\frac{1}{2}W}{v} \leqq t \leqq \infty$, with the additional restriction that $t \leqq t_0$. That is to say, as long as the rain lasts, equation (10) applies. As $t \rightarrow \infty$, y approaches the limit WLr. It should be noted that equation (4) has this same limit as $t \rightarrow \infty$. In other words, for equations (4) and (10), if the rain lasts sufficiently long a steady state will eventually be reached when there is just as much water flowing away from the area as there is rain falling upon it; this is just as would be expected, and moreover the discharge in this steady state is not dependent on whether overland travel is considered or neglected. From the nature of the problem, and without going into any involved mathematics, it is clear that the discharge at the time of a steady state is independent of the size and shape of the drainage area.

When the time $t<\frac{L}{v}$, the discharge, y, will not be given by the above integral, but will be the sum of the discharges from a rectangle whose width is W and length is $tv-\frac{1}{2}W$ and from a triangle whose base is W and altitude is $\frac{1}{2}W$. (See fig. 3.) The discharge from the rectangle is

$$\int_0^{tv-\frac{1}{2}W}\int_{-\frac{1}{2}W}^{\frac{1}{2}W}\left\{1-e^{-\frac{1}{c}\left(t-\frac{x+w}{v}\right)}\right\}dwdx,$$

and the discharge from the triangle is

$$\int_{tv-\frac{1}{2}W}^{tv}\int_{-(tv-x)}^{(tv-x)}\left\{1-e^{-\frac{1}{c}\left(t-\frac{x+w}{v}\right)}\right\}dwdx.$$

On performing the above integration, substituting the indicated limits, simplifying and adding, we get the equation:

$$y=r\left[Wtv-Wcv-\tfrac{1}{4}W^2+2c^2v^2e^{-\frac{t}{c}}\left(e^{\frac{W}{2cv}}-1\right)\right]. \qquad (11)$$

Equation (11) holds only on the range $\frac{W}{2v} \leqq t \leqq \frac{L}{v}$ with the restriction that $t \leqq t_0$. When overland travel was neglected, the curve, y, representing the discharge from the time the rain begins to the time after the rain stops when the discharge has receded to its value when the rain began, was a continuous curve whose first derivative with respect to time, dy/dt, was continuous and whose second derivative with respect to time, $\frac{d^2y}{dt^2}$, was also continuous except at three points, viz., $t=Lv$, $t=t_0$, and $t=t_0+L/v$. The effect of these three points of discontinuity in $\frac{d^2y}{dt^2}$ was to divide the curve for y into four sections represented by equations (3), (4), (6), and (5), respectively. Figure 4 illustrates the integral of the discharge curve. Figures (5), (6), and (7) illustrate a theoretical discharge curve, its first derivative, and its second derivative for a rectangular drainage area where overland travel is neglected. Figure (7) clearly shows the three points of discontinuity. As the derivative of a function is discontinuous at every point where the function is discontinuous, it follows that the third derivative of y with respect to t will also be discontinuous at these three points. Now when the question of overland travel is considered, as above, then not only both y and dy/dt

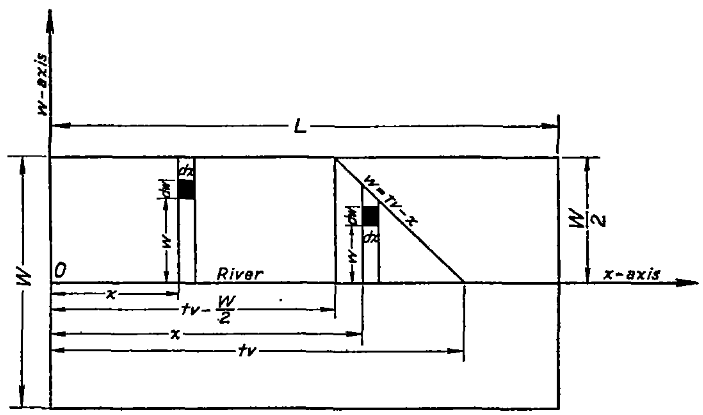


FIGURE 3.

but also $\frac{d^2y}{dt^2}$ are everywhere continuous. However, $\frac{d^3y}{dt^3}$ is continuous except at, not three points, but seven points, and the y-curve is thus divided into eight sections. The following table gives the range of t for each of the eight sections and the number of double integrals required to obtain y:

Range of t:	Number of double integrals
$0 \leqq t \leqq \frac{W}{2v}$	1
$\frac{W}{2v} \leqq t \leqq \frac{L}{v}$	2
$\frac{L}{v} \leqq t \leqq \frac{L+\frac{1}{2}W}{v}$	2
$\frac{L+\frac{1}{2}W}{v} \leqq t \leqq t_0$	1
$t_0 \leqq t \leqq t_0+\frac{W}{2v}$	3
$t_0+\frac{W}{2v} \leqq t \leqq t_0+\frac{L}{v}$	4
$t_0+\frac{L}{v} \leqq t \leqq t_0+\frac{L+\frac{1}{2}W}{v}$	3
$t_0+\frac{L+\frac{1}{2}W}{v} \leqq t \leqq \infty$	1

The foregoing table is based on the assumption that $t_0 \geqq (L+\frac{1}{2}W)/v$. Clearly t_0, the duration of the rain, may be less than $(L+\frac{1}{2}W)/v$. In such a case the above table would have to be modified, and it may happen that as many as five double integrals would be required to obtain y.

Equations for y as a function of t on each of the above ranges will not be given here; should they be desired, their derivation is perfectly straightforward, though tedious, as the above two equations show. As the main purpose of these papers is to devise a scheme for predicting flood crests, only one more equation for these ranges will be obtained, viz., that for the range $t_0+W/2v \leqq t \leqq t_0+L/v$, as this range contains the time of the crest.

It follows by reasoning similar to that given in the first article, together with that already given here, that the discharge on this last-mentioned range is given by the sum of (1) the discharge of a rectangle of width W and length $x_0-\frac{1}{2}W$ where the rate of run-off is decreasing; (2) the discharge of a triangle of base W and altitude $\frac{1}{2}W$ where the rate of run-off is decreasing; (3) the discharge of *two* triangles, each of base $\frac{1}{2}W$ and altitude $\frac{1}{2}W$, adjacent to the triangle of (2) and where the rate of run-off is increasing; and (4) the discharge of a rectangle of width W and length $L-x_0$., where the rate of run-off is increasing. The discharge is, therefore, the following sum of four double integrals:

$$
\begin{aligned}
y=&\int_0^{x_0-\frac{1}{2}W}\int_{-\frac{1}{2}W}^{\frac{1}{2}W} r\left(1-e^{-\frac{t_0}{c}}\right)e^{-\frac{1}{c}\left(t-t_0-\frac{x+w}{v}\right)}dwdx+\\
&\int_{x_0-\frac{1}{2}W}^{x_0}\int_{-(x_0-x)}^{x_0-x} r\left(1-e^{-\frac{t_0}{c}}\right)e^{-\frac{1}{c}\left(t-t_0-\frac{x+w}{v}\right)}dwdx+\\
&2\int_{x_0-\frac{1}{2}W}^{x_0}\int_{x_0-x}^{\frac{1}{2}W} r\left\{1-e^{-\frac{1}{c}\left(t-\frac{x+w}{v}\right)}\right\}dwdx+\\
&\int_{x_0}^{L}\int_{-\frac{1}{2}W}^{\frac{1}{2}W} r\left\{1-e^{-\frac{1}{c}\left(t-\frac{x+w}{v}\right)}\right\}dwdx.
\end{aligned}
$$

On performing the above integrations, substituting the indicated limits, simplifying, remembering here that $x_0=v(t-t_0)$, and adding, we get the equation:

$$
\begin{aligned}
y=2r\Bigg[&c^2v^2\left(e^{\frac{t_0}{c}}-1\right)\left\{e^{-\frac{t_0}{c}}-e^{-\frac{1}{c}\left(t_0+\frac{W}{2v}\right)}-e^{-\frac{1}{c}\left(t-\frac{W}{2v}\right)}+e^{-\frac{t}{c}}\right\}+\\
&cv\left(1-e^{-\frac{t_0}{c}}\right)\left\{\tfrac{1}{2}W-cv+cve^{-\frac{W}{2cv}}\right\}+\\
&\tfrac{1}{8}W^2+cve^{-\frac{t_0}{c}}\left\{\tfrac{1}{2}W+cv-cve^{\frac{W}{2cv}}\right\}+\\
&\tfrac{1}{2}WL-\tfrac{1}{2}Wv(t-t_0)-c^2v^2\left(e^{-\frac{1}{c}\left(t-\frac{L}{v}\right)}-e^{-\frac{t_0}{c}}\right)\left(e^{\frac{W}{2cv}}-1\right)\Bigg]\cdot
\end{aligned}
\tag{12}
$$

We can find the time of the maximum discharge by differentiating equation (12) with respect to t, and setting this derivative equal to zero. On doing this and simplifying, the following equation is obtained which gives the time of the crest:

$$t_c=c\log\left(e^{\frac{L}{cv}}+e^{\frac{t_0}{c}}-1\right)+c\log\frac{2cv}{W}\left(e^{\frac{W}{2cv}}-1\right)\cdot \tag{13}$$

Equation (13) agrees exactly with equation (7) except for the addition of the term $c\log\frac{2cv}{W}\left(e^{\frac{W}{2cv}}-1\right)\cdot$ It will now be shown that in all practical cases equation (7) gives t_c with sufficient accuracy:

Consider L to be 50 miles (for a value of L this small, it is doubtful if it would be practicable to make flood crest forecasts). L is purposely taken this small because as L increases the effect of the term $c\log\frac{2cv}{W}\left(e^{\frac{W}{2cv}}-1\right)$ on t_c becomes less and less. Take W as 2 miles; this means that the greatest distance that water would have to travel overland is 1 mile. It is believed that this value is greater than is usually found in nature. It may be well to point out here that the fact that a stream has tributaries, thus increasing the width of the *drainage basin*, will not increase W; $W/2$ is the greatest distance that water would have to travel *overland* to reach *some* tributary (i. e. a permanent stream channel). Except on extremely flat land, $W/2$ would not ordinarily exceed 1 mile; a glance at any topographic map will bear this out. Moreover, increasing the number of tributaries will have no effect on t_c; it will affect y_c but not t_c.

Consider now the constant c. This is a time constant, and when we say that a certain parcel of ground in a specified condition has a constant c equal to a certain number of hours (say 4 hours) we mean that at the end of 4 hours from the beginning of the rain the rate of run-off from the parcel of ground will be $\left(1-\frac{1}{e}\right)$ times the rate of rainfall which is falling upon the ground. Thus the time constant is the time in which the rate of run-off reaches $\left(1-\frac{1}{e}\right)=\frac{1.71828\ldots}{2.71828\ldots}=0.63212$ (approximately) of its final value (steady state). Clearly, the longer the time necessary for the rate of run-off to reach 0.63212 of the rate of rainfall, the greater is the amount of water which the soil retains; but the point to be emphasized is that we measure the capacity (dryness) of the soil in time (hours) and *not* in volume as might be supposed. It is believed that c is always greater than 1 hour. For soils covered with growing vegetation, 24 hours would probably be a good average value of c.

When streams are low, v may be as small as $\frac{1}{4}$ mile per hour or even less; but when a stream rises sufficiently to cause a flood, v will usually be much greater. Summing up, we can assume that in practice $\frac{2cv}{W}>1$, $c>1$ and $v>1$. Under these conditions it can be shown that $\log\frac{2cv}{W}\left(e^{\frac{W}{2cv}}-1\right)$ is everywhere positive but less than 0.541. It has the value 0.541 when $2cv/W=1$, and approaches zero very rapidly when $2cv/W$ is greater than 1. Of course, it is necessary to multiply the small quantity $\log(2cv/W)\left(e^{\frac{2cv}{W}}-1\right)$ by c, and the product might be appreciable. Accordingly the following table has been prepared showing the relative values of the two terms on

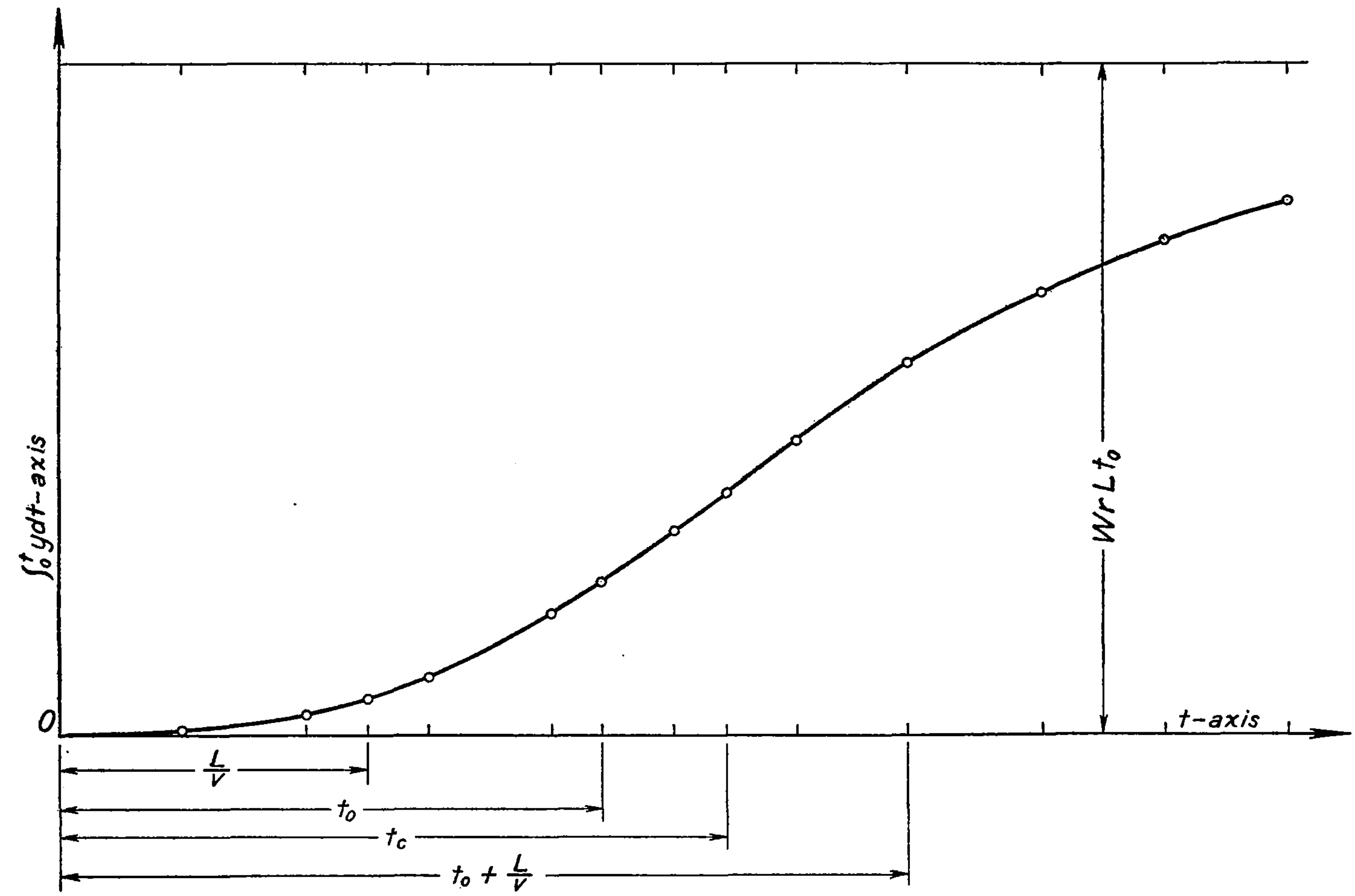
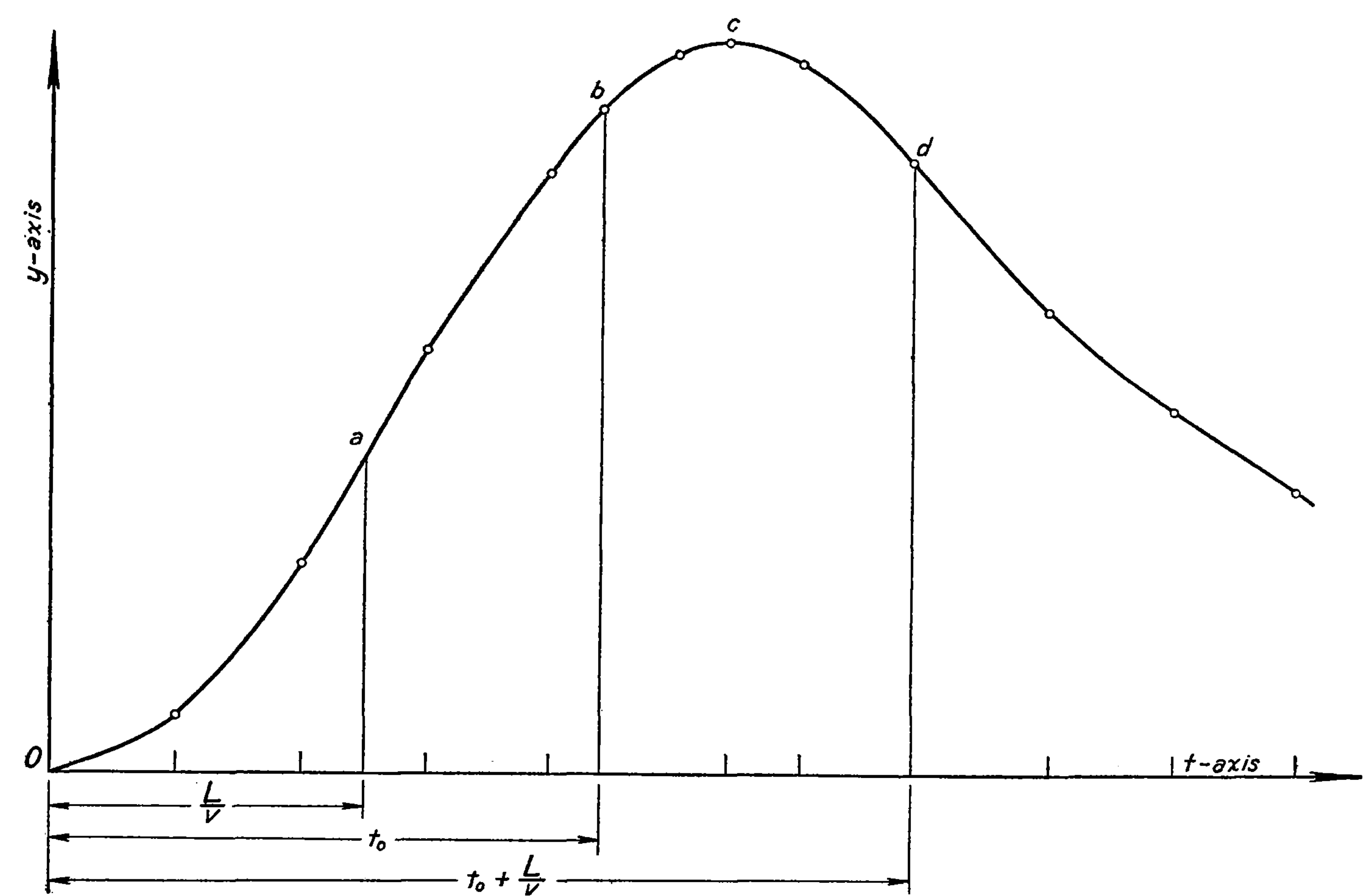

$\int_0^t ydt$-axis
$WrLt_o$
0
t-axis
$\frac{L}{V}$
t_o
t_c
$t_o + \frac{L}{V}$

FIGURE 4.


y-axis
a
b
c
d
0
t-axis
$\frac{L}{V}$
t_o
$t_o + \frac{L}{V}$

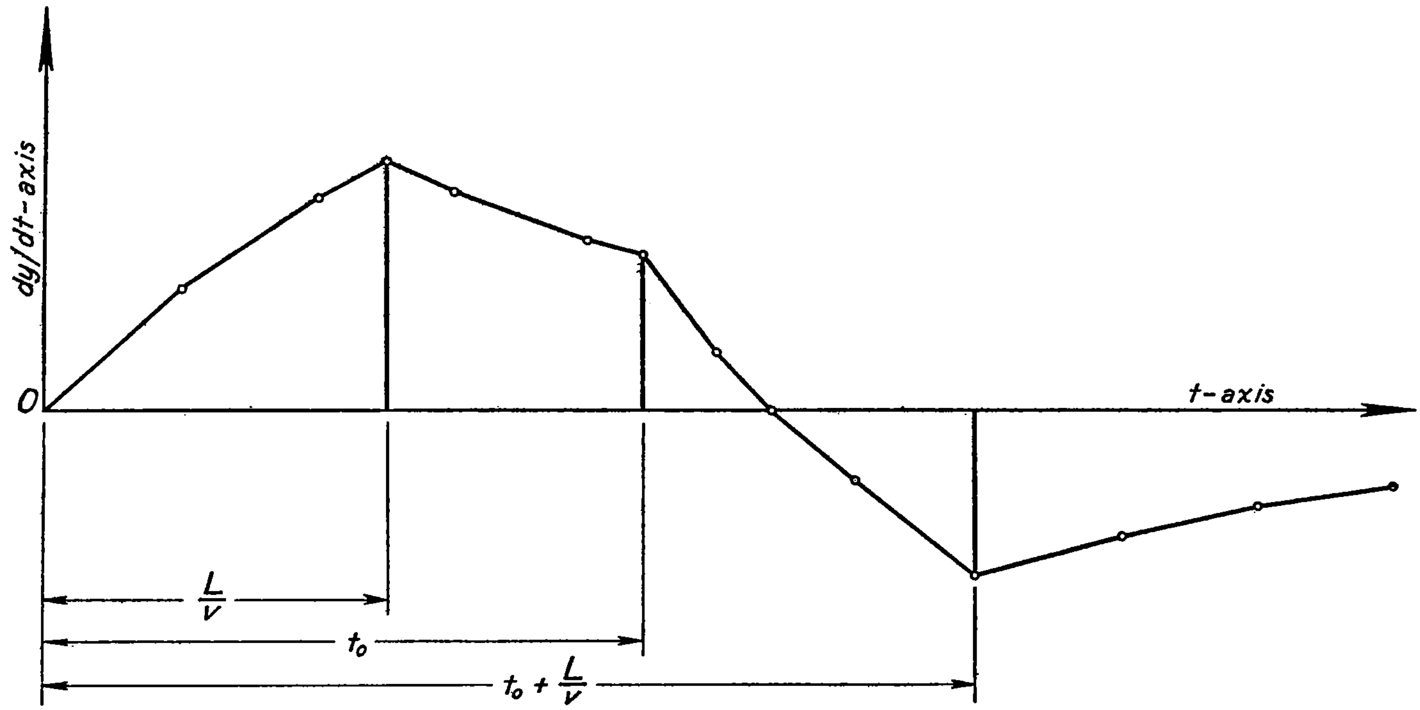


FIGURE 6.

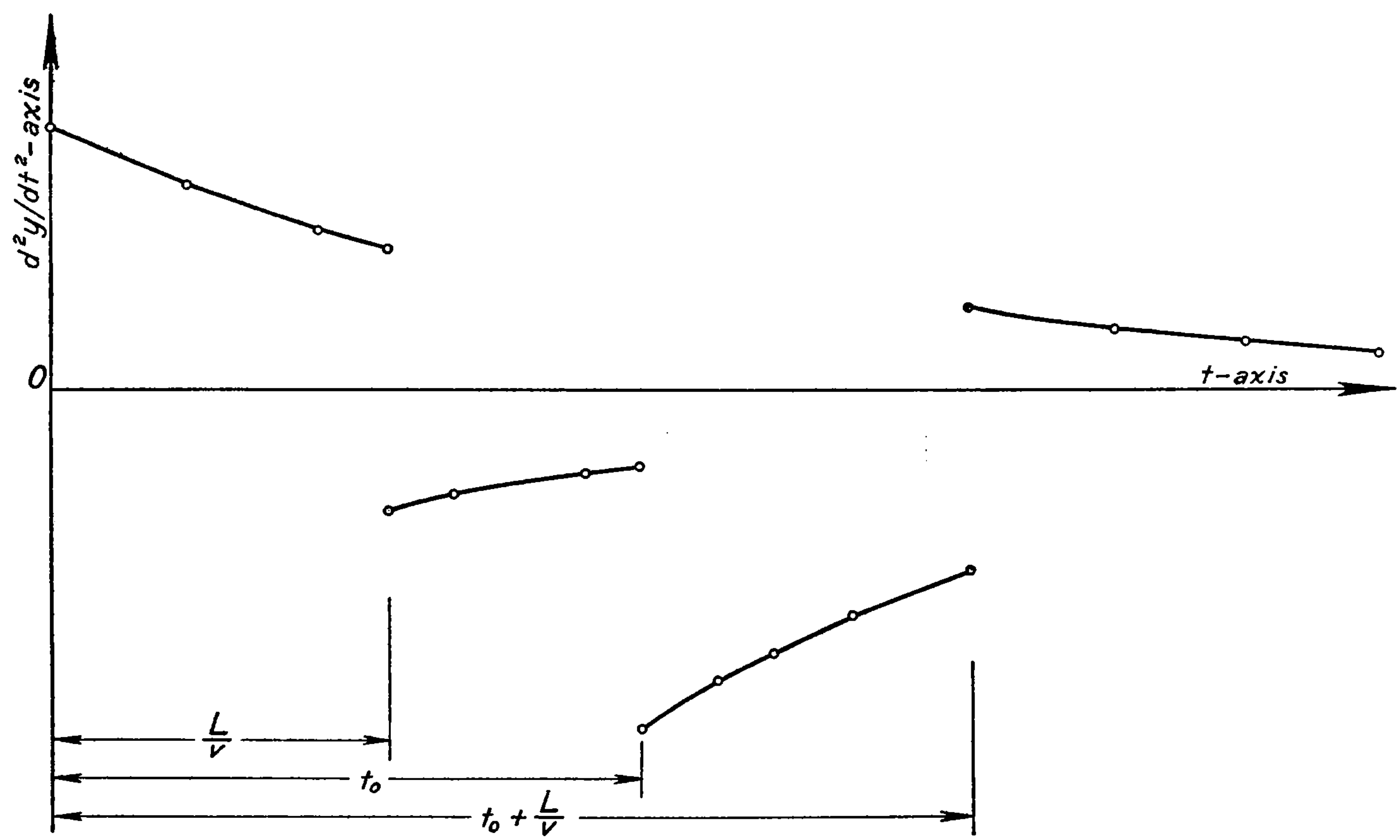


FIGURE 7.

the right-hand side of equation (13) for selected values of t_0, v, and c. In all cases $L=50$ miles and $W=2$ miles.

t_0	v	c	$c \log\left(e^{\frac{L}{cv}}+e^{\frac{t_0}{c}}-1\right)$	$c \log\frac{2cv}{W}\left(e^{\frac{W}{2cv}}-1\right)$
Hours	Miles per hour	Hours	Hours	Hours
18	3	4	20.14	0.17
15	4	4	16.65	.13
15	4	6	17.73	.13
15	4	8	18.65	.14
15	4	10	19.42	.16
13	4	10	18.18	.16
17	3	4	19.58	.17
17	4	4	18.08	.13

In all of the above cases the second term of equation (13) is less than 1 percent of the first term. The error caused by dropping this term is therefore negligible. Of course as c increases, the second term increases faster than the first, but a larger value of L would more than offset this. In all of the above cases, t_c is very small, mainly because of the small value of L; and for these small values of t_c, it would be impracticable to make flood crest warnings. It is doubtful whether it would be practicable to issue flood crest warnings when L is less than 125 miles. We conclude, therefore, that the error caused by dropping the second term in equation (13) is negligible in all practical cases.

The question now arises as to what will be the error in y_c caused by dropping the second term of equation (13). Clearly, if a stream has many tributaries, the discharge at the gaging station may be considered as the sum of the discharges from each tributary; each such discharge is given by either equation (6) or (12), depending upon how much accuracy is desired. Thus the more tributaries a stream has, the less the error involved by using equations (6), (7), and (8) in place of the more accurate equations (12) and (13).

In the following table there is listed the maximum discharge from a drainage area where $W=2$ miles, $L=50$ miles, and $r=0.20$ inch per hour.

t_0	v	c	y_c	y_c
Hours	Miles per hour	Hours	Computed from equations (6), (7), and (8) (mile-inches per hour)	Computed from equations (12) and (13) (mile-inches per hour)
18	3	4	17.43	17.44
15	4	4	17.36	17.35
15	4	6	15.63	15.62
15	4	8	14.16	14.15
15	4	10	12.93	12.87
13	4	10	11.71	11.65
17	3	4	16.90	16.90
17	4	4	18.27	18.21

While there is a systematic error in t_c when overland travel is neglected, the above table shows that the error in y_c, caused by entirely neglecting overland travel, is negligible. The actual error in y_c is even less in practical cases, because the more tributaries a stream has, the less the effect of overland travel on y_c.

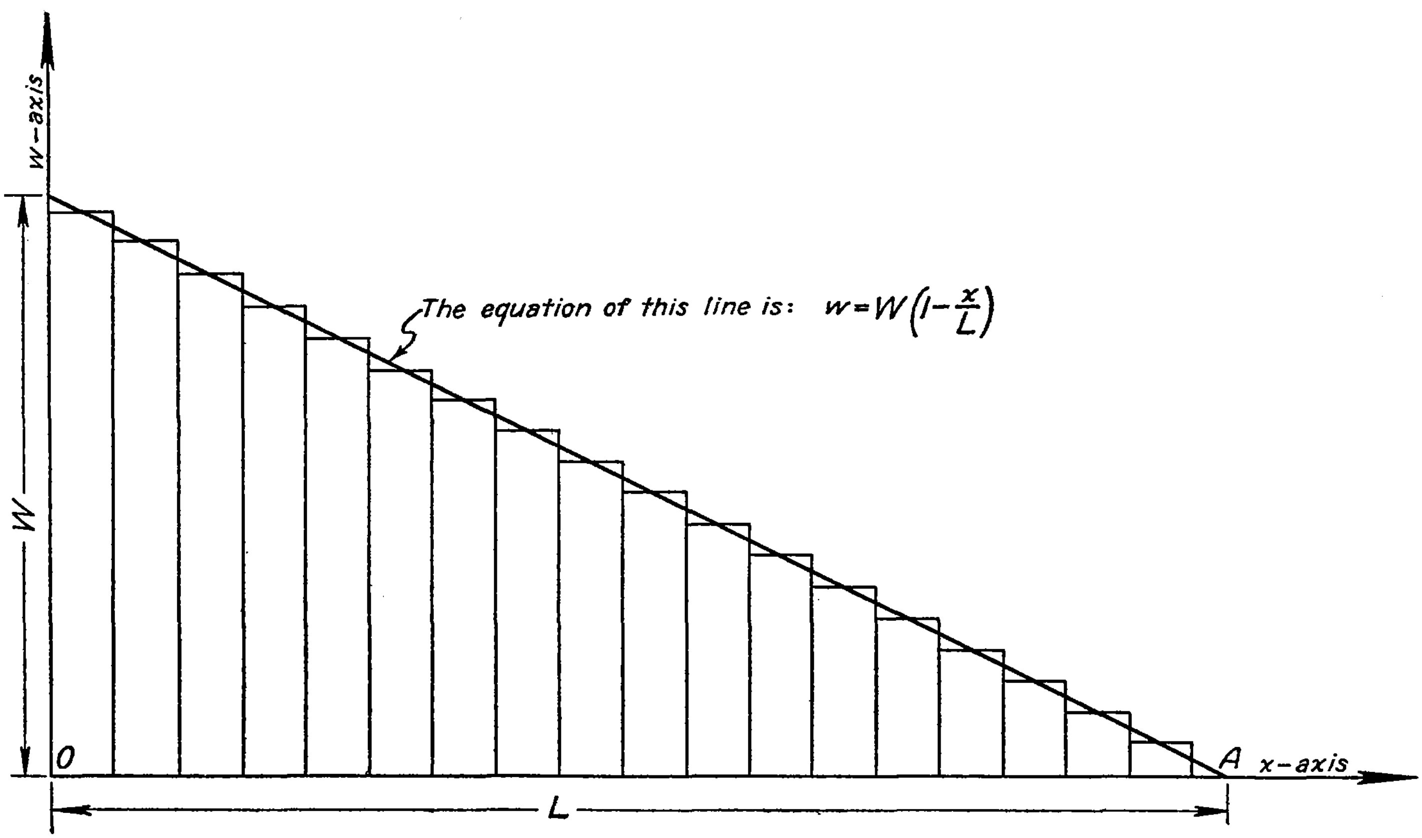


FIGURE 8.

SECTION 2: THE DISCHARGE FROM A TRIANGLE

Suppose that the histogram of a drainage area has the shape of a triangle. (See fig. 8.) Let the length of the base of the triangle be the length of the main stream of the river, and call this length L. Let W be the extreme width of the histogram. Then the width W at distance x above the gaging station will be $(W/L)x$. For the sake of brevity put $W/L=a$. Suppose that $t_0>L/v$. Then it follows, by reasoning similar to that in the first paper, that the discharge on the range $t_0 \leqq t \leqq t_0 + L/v$ from such a drainage area is given by

$$y = \int_{x_0}^{L} ar\left(1 - e^{-\frac{1}{c}\left(t - \frac{x}{v}\right)}\right) x dx + \int_0^{x_0} ar\left(1 - e^{-\frac{t_0}{c}}\right) e^{-\frac{1}{c}\left(t - t_0 - \frac{x}{v}\right)} x dx$$

and on performing the integration, substituting the indicated limits and simplifying, recalling here that $x_0 = v(t - t_0)$, we obtain the equation:

$$y = ar\left[\frac{L^2}{2} - \frac{(t - t_0)^2}{2} v^2 + c^2 v^2 \left\{\frac{1}{c}(t - t_0) - 1 + e^{-\frac{t}{c}}\left(e^{\frac{L}{cv}}\left\{1 - \frac{L}{cv}\right\} + e^{\frac{t_0}{c}} - 1\right)\right\}\right]. \tag{14}$$

The time of maximum discharge, t_c, may be found by differentiating equation (14) with respect to time, and setting this derivative equal to zero. The first derivative is easily found to be

$$\frac{dy}{dt} = arv^2\left[t_0 - t + c\left\{1 - e^{-\frac{t}{c}}\left(e^{\frac{L}{cv}}\left\{1 - \frac{L}{cv}\right\} + e^{\frac{t_0}{c}} - 1\right)\right\}\right],$$

and on setting this equal to zero we replace t by t_c and obtain

$$t_0 - t_c + c\left\{1 - e^{-\frac{t_0}{c}}\left(e^{\frac{L}{cv}}\left\{1 - \frac{L}{cv}\right\} + e^{\frac{t_0}{c}} - 1\right)\right\} = 0;$$

if the terms be rearranged, we finally have the equation

$$\left(1 + \frac{t_0}{c}\right) e^{\frac{t_0}{c}} - \frac{t_c}{c} e^{\frac{t_0}{c}} = e^{\frac{L}{cv}}\left\{1 - \frac{L}{cv}\right\} + e^{\frac{t_0}{c}} - 1. \tag{15}$$

The time of the crest is the time, t_c, which satisfies equation (15). Equation (15) is of the form $Ae^{\xi} - \xi e^{\xi} = B$. Equations of this type cannot be solved explicitly in terms of elementary functions. Hence it is necessary to solve equation (15) by numerical methods, or by a nomogram.

Equation (14) was derived on the assumption that $t_0 > L/v$. Suppose $t_0 < L/v$. Then it follows, by reasoning similar to that given in the first paper in the development of equation (9), that the discharge is given by

$$y = \int_{x_0}^{tv} ar\left(1 - e^{-\frac{1}{c}\left(t - \frac{x}{v}\right)}\right) x dx + \int_0^{x_0} ar\left(1 - e^{-\frac{t_0}{c}}\right) e^{-\frac{1}{c}\left(t - t_0 - \frac{x}{v}\right)} x dx.$$

On performing the integration, substituting the indicated limits and simplifying, we obtain

$$y = arv^2\left[tt_0 - \frac{t_0^2}{2} - c^2\left\{\frac{t_0}{c} - e^{-\frac{t}{c}}\left(e^{\frac{t_0}{c}} - 1\right)\right\}\right]. \tag{16}$$

Equation (16) holds on the range $t_0 \leqq t \leqq L/v$. It can be shown that equation (16) does not have a maximum on the range for which it holds. Hence equation (14) always has a maximum regardless of whether $t_0 \lesseqgtr L/v$.

Suppose now that the gaging station is at the point A rather than at the point 0 in figure 8, i. e., suppose the river flows to the right instead of to the left. In other words, consider the case where the width of the drainage area decreases with increasing distance above the gaging station, rather than increases with increasing distance as formerly. Then w is not given by $(W/L)x$, but by the equation $w = W - (W/L)x = W - ax$. Suppose $t_0 > L/v$; then on the range $t_0 \leqq t \leqq t_0 + L/v$ the discharge is given by

$$y = \int_{x_0}^{L} (W - ax) r\left[1 - e^{-\frac{1}{c}\left(t - \frac{x}{v}\right)}\right] dx + \int_0^{x_0} (W - ax) r\left(1 - e^{-\frac{t_0}{c}}\right) e^{-\frac{1}{c}\left(t - t_0 - \frac{x}{v}\right)} dx;$$

and on performing the indicated operations and simplifying, we get

$$y = r\left[WL - Wv(t - t_0) + Wcve^{-t/c}\left\{e^{t/c} + 1 - e^{\frac{L}{cv}} - e^{\frac{t_0}{c}}\right\} - \frac{aL^2}{2} + \frac{av^2(t - t_0)^2}{2} - ac^2v^2\left\{\frac{1}{c}(t - t_0) - 1 + e^{-t/c}\left(e^{\frac{L}{cv}}\left\{1 - \frac{L}{cv}\right\} + e^{\frac{t_0}{c}} - 1\right)\right\}\right]. \tag{17}$$

The time of the maximum discharge for equation (17) can be found in the usual manner.

If $t_0 < L/v$, then on the range $t_0 \leqq t \leqq L/v$ the discharge is given by

$$y = \int_{x_0}^{tv} (W - ax) r\left[1 - e^{-\frac{1}{c}\left(t - \frac{x}{v}\right)}\right] dx + \int_0^{x_0} (W - ax) r\left(1 - e^{-\frac{t_0}{c}}\right) e^{-\frac{1}{c}\left(t - t_0 - \frac{x}{v}\right)} dx.$$

On carrying out the indicated operations we get

$$y = Wrv\left[t_0 + c\left\{e^{-\frac{t}{c}} - e^{-\frac{1}{c}(t - t_0)}\right\}\right] - arv^2\left[tt_0 - \frac{t_0^2}{2} - c^2\left\{\frac{t_0}{c} - e^{-\frac{t}{c}}\left(e^{\frac{t_0}{c}} - 1\right)\right\}\right]. \tag{18}$$

If we differentiate equation (18) with respect to time, t, and set the derivative equal to zero and solve the resulting equation for t_c we find that equation (18) has a maximum when

$$t_c = c \log\left[\left(e^{\frac{t_0}{c}} - 1\right)\left(\frac{W}{avt_0} + \frac{c}{t_0}\right)\right]. \tag{19}$$

Equations (9), (16), and (18) all hold on the range $t_0 \leqq t \leqq L/v$, but apply to differently shaped drainage areas. It should be noted that while equations (9) and (16) have no maxima on this range, equation (18) does. The important fact which equations (18) and (19) establish is that the crest, or maximum discharge, *may* occur in some drainage areas *before* any water from the upper part of the drainage area has reached the gage. This feature will be discussed further in section 4 of this paper.

The first derivative of equation (16) is easily found to be $dy/dt = arv^2\left[t_0 - ce^{-t/c}\left(e^{\frac{t_0}{c}} - 1\right)\right]$, and the second derivative is $d^2y/dt^2 = arv^2\left[e^{-t/c}\left(e^{\frac{t_0}{c}} - 1\right)\right]$. Suppose we equate the first derivative to zero, and call the particular value of t which satisfies the equation t_c; we can solve this equation for t_c; thus

$$t_c = c \log \frac{c}{t_0}\left(e^{\frac{t_0}{c}} - 1\right). \tag{20}$$

If we substitute this value of t_c in the second derivative, we get $d^2y/dt^2=arv^2c/t_0$, which is a positive quantity, hence equation (2) gives the value of a *minimum* for equation (16). Now from the physical nature of the problem, we should not expect equation (16) to have a proper minimum on the range for which it applies, viz., $t_0 \leqq t \leqq L/v$. It will now be shown that the value of t_c given by equation (20) is less than t_0. Write equation (2) in the form $\frac{t_0}{c}e^{\frac{t_c}{c}}=\left(e^{\frac{t_0}{c}}-1\right)$, and for brevity put $\frac{t_c}{c}=\eta$ and $\frac{t_c}{c}=\zeta$; then we get $e^{\zeta}=\frac{1}{\eta}(e^{\eta}-1)$. Now expand the exponentials in a series, thus:

$$1+\zeta+\zeta^2/2!+\zeta^3/3!+\zeta^4/4!+\ \ldots\ =$$

$$\frac{1}{\eta}\left(\eta+\frac{\eta^2}{2!}+\eta^3/3!+\eta^4/4!+\eta^5/5!+\ \ldots\ \right)$$

$$=1+\eta/2!+\eta^2/3!+\eta^3/4!+\ \ldots$$

If $\zeta \geqq \eta$, then each term of the second series, $\eta^n/(n+1)!$ is *less* than the corresponding term, $\zeta^n/n!$ of the first, hence in order that the equality may be satisfied, $\zeta<\eta$, i. e., $t_c<t_0$; and equation (16) has no minimum on the range for which it applies.

In the case of the rectangle it is always possible to find the time of the crest, t_c, explicitly. In the case of the triangle, the time of the crest can be expressed explicitly in some instances but not in all; thus the derivatives of equations (16) and (18) can be solved explicitly for t_c, while those of equations (14) and (17) cannot. We shall now consider drainage areas where in no case is it possible to express the time of the crest explicitly.

SECTION 3: THE DISCHARGE FROM AN ELLIPSE

Consider now the case when the histogram may be fitted by a semiellipse; if L be the length of the histogram and W the extreme height of the histogram, then the equation is $(w/W)^2+(2x/L-1)^2=1$. (See figure 9.) It follows readily that the width of the drainage area at distance x from the gaging station, with the origin at the latter, is given by the equation $w=W\sqrt{1-(2x/L-1)^2}$. Instead of obtaining at once the equation on whose range the crest occurs, as was done in section 2, consider the range $L/v \leqq t \leqq t_0$; this was the range for equation (4) in the first paper. On this range the discharge is given by the integral:

$$y=\int_0^L Wr\sqrt{1-(2x/L-1)^2}\left[1-e^{-\frac{1}{c}\left(t-\frac{x}{v}\right)}\right]dx.$$

As this integral cannot be evaluated in terms of elementary functions, each step in performing the above integration will be given in detail. First dividing by Wr, and then multiplying the expressions under the integral sign, we get:

$$y/Wr=\int_0^L \sqrt{1-(2x/L-1)^2}dx$$

$$-\int_0^L \sqrt{1-(2x/L-1)^2}e^{-\frac{1}{c}\left(t-\frac{x}{v}\right)}\,dx.$$

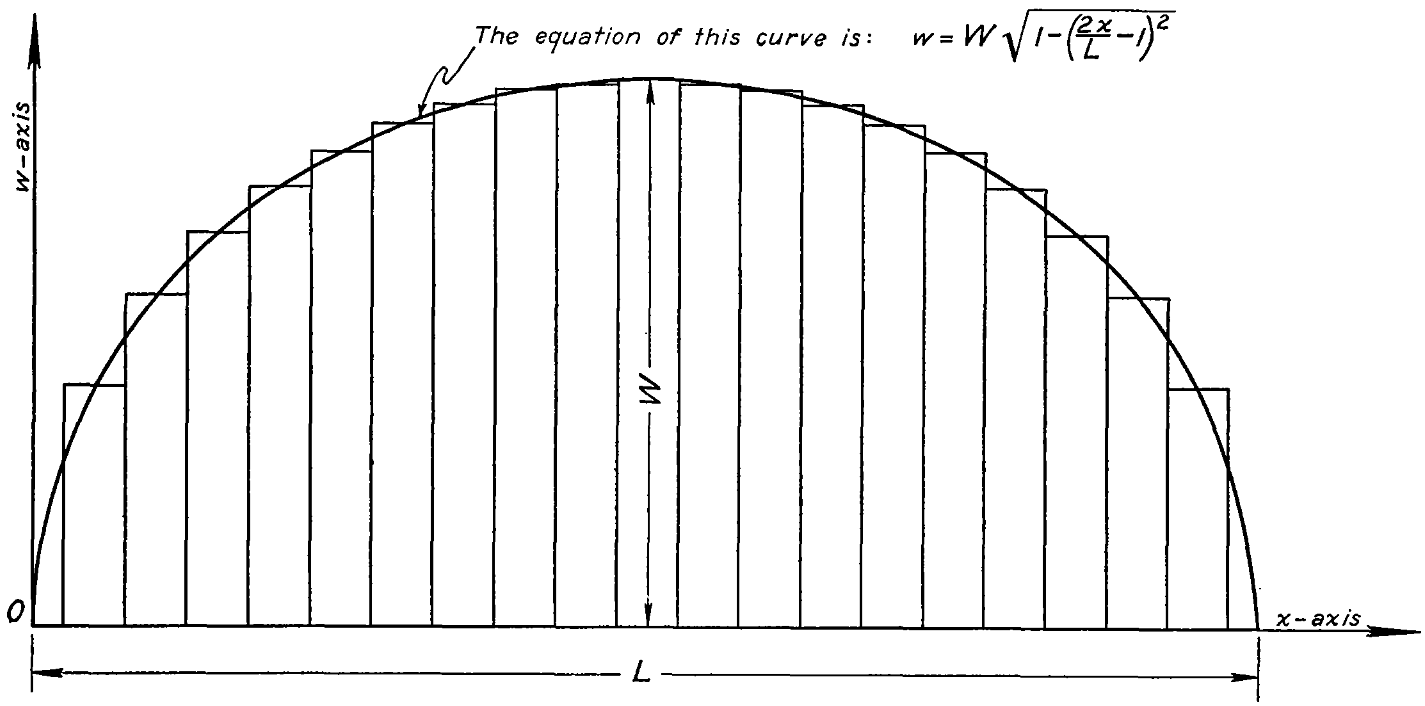


FIGURE 9.

The first integral in this last equation can be evaluated in terms of elementary functions, and is readily found to be

$$\frac{L}{2}\ \frac{1}{2}\left[(2x/L-1)\sqrt{1-(2x/L-1)^2}\right.$$

$$\left.+\text{arc sin }(2x/L-1)\right]_0^L=\frac{\pi L}{4}.$$

The second integral can be written in the form:

$$e^{-\frac{t}{c}}\int_0^L \sqrt{1-(2x/L-1)^2}e^{\frac{x}{cv}}dx.$$

Now replace x by $\frac{1}{2}L\theta+\frac{1}{2}L$; then when $x=0$, $\theta=-1$, and when $x=L$, $\theta=1$; also $dx=(L/2)\,d\theta$. Making these substitutions, we have

$$e^{-\frac{t}{c}}\int_{-1}^{+1}\sqrt{1-\theta^2}\,e^{\frac{L}{2cv}(\theta+1)}\,(L/2)d\theta$$
$$=\frac{1}{2}Le^{-\frac{1}{c}\left(t-\frac{L}{2v}\right)}\int_{-1}^{+1}\sqrt{1-\theta^2}\,e^{L\theta/2cv}\,d\theta$$
$$=\frac{1}{2}Le^{-\frac{1}{c}\left(t-\frac{L}{2v}\right)}\int_{-1}^{+1}\sqrt{1-\theta^2}\,e^{\frac{L}{2cv}\theta}\,d\theta$$
$$=\frac{1}{2}Le^{-\frac{1}{c}(t-L/2v)}\left(\frac{\pi}{L/2cv}\right)I_1(L/2cv)$$
$$=\pi cve^{-\frac{1}{c}(t-L/2v)}\,I_1(L/2cv),$$

where $I_1(L/2cv)$ is the *modified Bessel Function of the first kind of order one* with argument $L/2cv$. The discharge from the semiellipse on the range $L/v\leqq t\leqq t_0$ is therefore given by

$$y=\frac{1}{4}WLr\pi-Wrcv\pi e^{-\frac{1}{c}(t-L/2v)}\,I_1(L/2cv). \qquad (21)$$

Next consider the range $0\leqq t\leqq L/v$, where also $t\leqq t_0$; here the discharge is given by the integral:

$$y=\int_0^{tv}Wr\sqrt{1-(2x/L-1)^2}\left[1-e^{-\frac{1}{c}\left(t-\frac{x}{v}\right)}\right]dx.$$

In trying to evaluate this integral we divide the equation by Wr and multiply the expressions under the integral sign and get

$$y/Wr=\int_0^{tv}\sqrt{1-(2x/L-1)^2}dx$$
$$-\int_0^{tv}\sqrt{1-(2x/L-1)^2}e^{-\frac{1}{c}\left(t-\frac{x}{v}\right)}dx.$$

By evaluating the first integral, and making the substitution $x=(L/2)\,(\theta+1)$ in the second, we have

$$\frac{Y}{Wr}=\frac{L}{2}\cdot\frac{1}{2}\left[(2x/L-1)\sqrt{1-(2x/L-1)^2}+\text{arc sin}(2x/L-1)\right]_0^{tv}$$
$$-e^{-\frac{t}{c}}\int_{-1}^{2tv/L-1}\sqrt{1-\theta^2}e^{L(\theta+1)2cv}(L/2)d\theta.$$

On further simplification, we find that on the range under consideration the discharge is given by

$$y=WLr\pi/8+\tfrac{1}{4}WLr\{(2t/L-1)\sqrt{1-(2tv/L-1)^2}$$
$$+\text{arc sin }(2tv/L-1\}$$
$$-\frac{1}{2}WLre^{-\frac{1}{c}\left(t-t_0-L/2v\right)}\int_{-1}^{2tv/L-1}\sqrt{1-\theta^2}e^{L\theta/2cv}d\theta. \qquad (22)$$

The integral occurring in equation (22) cannot be evaluated explicitly in terms of any mathematical functions now known. It is necessary to use numerical methods to evaluate it.

Consider now the range $t_0+L/v\leqq t\leqq\infty$. It follows, by reasoning similar to that used in deriving equation (5) in the first paper, that on this range the discharge from a semiellipse is given by

$$y=\int_0^L Wr\sqrt{1-(2x/L-1)^2}e^{-\frac{1}{c}\left(t-t_0-\frac{x}{v}\right)}dx,$$

and clearly this equation can be written in the form

$$y/Wr=e^{-\frac{1}{c}\left(t-t_0\right)}\int_0^L\sqrt{1-(2x/L-1)^2}e^{\frac{x}{cv}}dx.$$

On making the same substitutions used in deriving equation (21), we obtain

$$y/Wr=e^{-\frac{1}{c}\left(t-t_0\right)}\int_{-1}^{+1}\sqrt{1-\theta^2}e^{L(\theta+1)/2cv}(L/2)d\theta$$
$$=\frac{1}{2}Le^{-\frac{1}{c}\left(t-t_0-L/2v\right)}\int_{-1}^{+1}\sqrt{1-\theta^2}e^{L\theta/2cv}d\theta$$
$$=\pi cve^{\frac{1}{c}\left(t-t_0-L/cv\right)}I_1(L/2cv),$$

where $I_1\,(L/2cv)$ has the same meaning as in equation (21). On multiplying by Wr we finally obtain

$$y=Wrcv\pi e^{-\frac{1}{c}\left(t-t_0-L/2v\right)}I_1(L/2cv). \qquad (23)$$

To complete the treatment of the semiellipse where $t_0>L/v$, it remains to consider the range $t_0\leqq t\leqq {}_0+L/v$. It is on this range that the crest occurs. It follows by reasoning similar to that used in deriving equation (6) that on this range the discharge from a semiellipse is given by

$$y=\int_{x_0}^L Wr\sqrt{1-(2x/L-1)^2}\left[1-e^{-\frac{1}{c}\left(t-\frac{x}{v}\right)}\right]dx$$
$$+\int_0^{x_0}Wr\sqrt{1-(2x/L-1)^2}\left(1-e^{-\frac{t_0}{c}}\right)e^{-\frac{1}{c}\left(t-t_0-\frac{x}{v}\right)}dx.$$

On dividing by Wr and multiplying the expressions under the integral signs, we get

$$y/Wr=\int_{x_0}^L\sqrt{1-(2x/L-1)^2}dx-e^{-\frac{tc}{c}}\int_{x_0}^L\sqrt{1-(2x/L-1)^2}e^{\frac{x}{cv}}dx$$
$$+e^{-\frac{1}{c}\left(t-t_0\right)}\int_0^{x_0}\sqrt{1-(2x/L-1)^2}e^{\frac{x}{cv}}dx$$
$$-e^{-\frac{t}{c}}\int_0^{x_0}\sqrt{1-(2x/L-1)^2}e^{\frac{x}{cv}}dx.$$

The first integral in this last equation can be evaluated in terms of elementary functions; the second and fourth can be combined, using the limits 0 and L. On doing this and making the same substitution used in deriving equation (21), we finally obtain

$$y/Wr=\frac{1}{4}\Big[\pi-(2x_0/L-1)\sqrt{1-(2x_0/L-1)^2}$$
$$-\text{arc sin }(2x_0/L-1)\Big]$$
$$-\pi cve^{-\frac{1}{c}\left(t-L/2v\right)}I_1(L/2cv)$$
$$+\frac{1}{2}Le^{-\frac{1}{c}\left(t-t_0-L/2v\right)}\int_{-1}^{2x_0/L-1}\sqrt{1-\theta^2}e^{L\theta/2cv}d\theta.$$

On multiplying by Wr and recalling that $x_0=v(t-t_0)$, we now get

$$y=Wr\Big\{\frac{1}{4}L[\pi-(2v\{t-t_0\}/L-1)\sqrt{1-(2v\{t-t_0\}/L-1)^2}$$
$$-\text{arc sin }(2v\{t-t_0\}/L-1)]-\pi cve^{-\frac{1}{c}\left(t-\frac{L}{3v}\right)}I_1(L/2cv)$$
$$+\frac{1}{2}Le^{-\frac{1}{c}\left(t-t_0-L/2v\right)}\int_{-1}^{2v(t-t_0)/L-1}\sqrt{1-\theta^2}e^{L\theta/2cv}d\theta\Big\}. \qquad (24)$$

Equation (22) expresses y as a function of t, but the function is not an elementary one; in fact it is not possible to evaluate the integral in this equation in terms of any known mathematical functions. In no one of the four ranges which have been considered is it possible—except for the particular values $t=0$, $t=\infty$, and also as will be pointed out presently $t=t_c$—to express y as a function of t by means of the elementary functions. In the cases of equations (21) and (23), y involves the modified Bessel function of the first kind of order one, while equations (22) and (24) are still more complex and involve a different and unnamed function. This is readily seen when we observe that in the development of these four equations the integrand encountered is the same; and, while in deriving equations (21) and (23) the upper limit is fixed, for the other two equations the upper limit is variable. Now it is an interesting fact that, while in general y is not an elementary function of t on the range $t_0\leqq t\leqq t_0+L/v$, for the particular value of t when the maximum discharge occurs y can be expressed in terms of elementary functions.

The maximum discharge, y_c, can be obtained from equation (24) as follows: Multiply both sides of equation (24) by $e^{t/c}$. Then differentiate both sides of the resulting equation with respect to t; recall here that the formula for differentiating under the integral sign is

$$\frac{d}{d\alpha}\int_a^b f(x,\alpha)dx=\int_a^b\frac{\partial f}{\partial\alpha}dx+f(b,\alpha)\frac{db}{d\alpha}-f(a,\alpha)\frac{da}{d\alpha}.$$

Then put $dy/dt=0$ and finally multiply by $ce^{-t/c}$. Then on collecting terms on the right, we have:

$$y_c=\tfrac{1}{4}WLr[\pi-(2v\{t_c-t_0\}/L-1)\sqrt{1-(2v\{t_c-t_0\}/L-1)^2}-\text{arc sin }(2v\{t_c-t_0\}/L-1)]. \tag{25}$$

No equations will be derived for the case when $t_0<\frac{L}{v}$. It is not likely that they would be of practical value; but if they should be desired, the reader should have no difficulty in obtaining them by using steps quite similar to those given above.

It will now be clear that the triangular and elliptically shaped drainage areas involve considerably more difficulties than the rectangular. For these reasons, and also because comparatively few watersheds would have a histogram whose shape is that of a simple geometric figure, no further equations will be given for geometric configurations. The difficulties mentioned in connection with the triangle and the ellipse should be kept in mind as one reads the remainder of this paper.

SECTION 4: THE DISCHARGE FROM ANY DRAINAGE AREA

In this section all of the formulas of both this article and the first article are summarized. This summary should be of much convenience to those who wish to apply the theory developed in this series of articles.

Instead of considering a drainage area whose histogram is of a special shape, as an ellipse or a triangle, or one to whose histogram a particular mathematical equation has been fitted, consider the more general case of a drainage area whose histogram may be represented by the function $w=W(x)$. When for $W(x)$ we substitute special functions we get special cases; thus, if $W(x)=W$, a constant, we have the case of a rectangle, if $W(x)=W-(W/L)x$, we have one of the triangular cases, and if $W(x)=W\sqrt{1-(2x/L-1)^2}$ we have the case of a semiellipse, all of which have already been discussed.

Let the greatest distance that water has to travel in order to reach the gage be L. Then the following mathematical restrictions will be placed on $W(x)$:

1. $W(x)$ shall be single-valued, except at points of discontinuity where it shall be two-valued;
2. $W(x)$ shall be everywhere finite (in other words, $W(x)$ is bounded);
3. $W(x)$ shall be everywhere positive, except possibly at $x=L$ where it may have the value zero;
4. $W(x)$ shall not have more than a finite number of discontinuities on a finite range.

Each of the above restrictions is justified by the physical nature of the problem. Thus a drainage area curve which is not single-valued, or is infinite, or is negative, would have no practical meaning. It may be thought that the last restriction is more general than is necessary practically, and that to say $W(x)$ shall be everywhere continuous would be sufficiently general. However, a mountain range, or a divide between two tributaries, within a drainage area, may cause a discontinuity in the drainage area curve. Thus, if a river has cut a gorge through a range of mountains, and if the tributaries downstream from the gorge and range are of such nature that the range itself corresponds to an equal water travel line, then there will be a discontinuity in the drainage area curve.

It follows, by reasoning similar to that used in deriving equation (3) in the first paper, that on the range $0\lesseqgtr t\lesseqgtr L/v$ where $t\lesseqgtr t_0$ the discharge from a drainage area whose histogram is represented by $w=W(x)$ is given by

$$\begin{aligned}y&=r\int_0^{tv}W(x)\left[1-e^{-\frac{1}{c}\left(t-\frac{x}{v}\right)}\right]dx\\&=r\left\{\int_0^{tv}W(x)dx-e^{-\frac{t}{c}}\int_0^{tv}W(x)e^{\frac{x}{cv}}dx\right\}.\end{aligned} \tag{B-3}$$

We may say that the generalized form of equation (3) is equation (B-3). When $t=0$ then $y=0$, and when $t=L/v$ equation (B-3) takes the form:

$$y=r\left\{\int_0^L W(x)dx-e^{-\frac{L}{cv}}\int_0^L W(x)e^{\frac{x}{cv}}dx\right\}, \tag{B-3a}$$

which is the generalized form of equation (3a).

On the range $L/v\lesseqgtr t\lesseqgtr t_0$ the discharge is given by

$$\begin{aligned}y&=r\int_0^L W(x)\left[1-e^{-\frac{1}{c}\left(t-\frac{x}{v}\right)}\right]dx\\&=r\left\{\int_0^L W(x)dx-e^{-\frac{t}{c}}\int_0^L W(x)e^{\frac{x}{cv}}dx\right\},\end{aligned} \tag{B-4}$$

which is the generalized form of equation (4). Since $\int_0^L W(x)dx$ is the area of the drainage basin, and since as $t\to\infty$, $e^{-t/c}\to 0$, equation (B-4) shows that if the rain continues sufficiently long the discharge approaches the product $Ar\equiv r\int_0^L W(x)dx$, the area of the watershed times the rate of rainfall, as a limit. Equation (1) shows that if the rain continues sufficiently long, then the *volume of rate of run-off* also approaches Ar as a limit. However, the *volume of rate of run-off* approaches the limit Ar more quickly than does the *discharge*, because in equation (1) the coefficient of $e^{-t/c}$ is $\int_0^L W(x)dx=A$, while in equation (B-4) the coefficient of $e^{-t/c}$ is $\int_0^L W(x)e^{\frac{x}{cv}}dx$, and

$$\int_0^L W(x)e^{\frac{x}{cv}}dx>\int_0^L W(x)dx.$$

On the range $t_0+L/v \leqq t \leqq \infty$, the discharge is given by the generalized form of equation (5), viz,

$$y=r\int_0^L W(x)e^{-\frac{1}{c}\left(t-x/v\right)}\left(1-e^{-t_0/c}\right)dx$$
$$=re^{-\frac{t}{c}}\left(e^{\frac{t_0}{c}}-1\right)\int_0^L W(x)e^{\frac{x}{cv}}dx. \qquad (B\text{-}5)$$

As $t\rightarrow\infty$, the discharge as given by equation (B-5) approaches the limit zero. Likewise equation (2) shows that as $t\rightarrow\infty$, then the volume of rate of run-off approaches zero as a limit. As in the case of equations (1) and (B-4), equation (2) approaches the limit 0 more quickly than equation (B-5), and for the same reason.

When $t=L/v$, equation (B-4) takes the form of equation (B–3a); and when $t=t_0$, equation (B-4) becomes

$$y=r\left\{\int_0^L W(x)dx-e^{-\frac{t_0}{c}}\int_0^L W(x)e^{\frac{x}{cv}}dx\right\}. \qquad \text{(B–4a)}$$

When $t=t_0+L/v$, equation (B-5) takes the form:

$$y=re^{-\frac{L}{cv}}\left(1-e^{-t_0/c}\right)\int_0^L W(x)e^{\frac{x}{cv}}dx. \qquad \text{(B–5a)}$$

Equations (B–4a) and (B–5a) are the generalized forms of equations (4a) and (5a).

On the range $t_0 \leqq t \leqq t_0+L/v$ where $t_0>L/v$, the discharge is given by [recall here that $v(t-t_0)=x_0$]

$$y=r\int_{x_0}^L W(x)\left[1-e^{-\frac{1}{c}\left(t-\frac{x}{v}\right)}\right]dx$$
$$+r\int_0^{x_0} W(x)e^{-\frac{1}{c}\left(t-\frac{x}{v}\right)}\left(1-e^{-\frac{t_0}{c}}\right)dx.$$

On multiplying the expressions under the integral sign and dividing by r, we get

$$y/r=\int_{x_0}^L W(x)dx-e^{-t/c}\int_{x_0}^L W(x)e^{\frac{x}{cv}}dx-e^{-t/c}\int_0^{x_0} W(x)e^{\frac{x}{cv}}dx$$
$$+e^{-\frac{1}{c}(t-t_0)}\int_0^{x_0} W(x)e^{\frac{x}{cv}}dx.$$

In this last equation the second and third integrals can be combined, using the limits *0* and *L*, whence we finally obtain

$$y=r\left\{\int_{x_0}^L W(x)dx-e^{-\frac{t}{c}}\int_0^L W(x)e^{\frac{x}{cv}}dx\right.$$
$$\left.+e^{-\frac{1}{c}(t-t_0)}\int_0^{x_0} W(x)e^{\frac{x}{cv}}dx\right\}. \qquad \text{(B–6)}$$

When $t=t_0$ then equation (B–6) reduces to equation (B–4a); and when $t=t_0+L/v$, equation (B–6) becomes equation (B–5a).

By applying the rule for differentiating under the integral sign, we find the derivative of equation (B–6) to be

$$dy/dt=r\left\{-vW(tv-t_0v)+\frac{1}{c}e^{-\frac{t}{c}}\int_0^L W(x)e^{\frac{x}{cv}}dx\right.$$
$$\left.-\frac{1}{c}e^{-\frac{1}{c}(t-t_0)}\int_0^{tv-t_0v} W(x)e^{\frac{x}{cv}}dx+ve^{-\frac{1}{c}(t-t_0)}W(tv-t_0v)e^{\frac{1}{c}(t-t_0)}\right\}.$$

Clearly the first and last terms on the right cancel, whence the above simplifies to

$$\frac{dy}{dt}=\frac{r}{c}e^{-\frac{t}{c}}\left\{\int_0^L W(x)e^{\frac{x}{cv}}dx-e^{\frac{t_0}{c}}\int_0^{tv-t_0v} W(x)e^{\frac{x}{cv}}dx\right\}.$$

Then on equating dy/dt to zero, we get

$$e^{\frac{t_0}{c}}\int_0^{t_cv-t_0v} W(x)e^{\frac{x}{cv}}dx=\int_0^L W(x)e^{\frac{x}{cv}}dx. \qquad \text{(B–7)}$$

Equation (B–7) is not a generalized form of equation (7); but the time of the crest, t_c, is given by the solution of equation (B–7).

The maximum discharge can be obtained by multiplying both sides of equation (B–6) by $e^{t/c}$, differentiating the resulting equation with respect to t, putting $dy/dt=0$, and finally solving for y_c by multiplying by $\frac{1}{c}e^{-t/c}$. On carrying out these steps we get

$$y_c=r\int_{v(t_c-t_0)}^L W(x)dx, \qquad \text{(B–8)}$$

which is the generalized form of equation (8). It should be noted that in the first paper equation (8) could have been derived from equation (6) by the procedure here suggested, instead of the more straightforward method used there; and moreover that equation (8) can be written in the form:

$$y_c=Wr\int_{v(t_c-t_0)}^L dx.$$

Equation (B–8) is an exceptionally neat way in which to express the maximum discharge, and further comments will be made on it later.

Equations (B–6), (B–7), and (B–8) were derived on the assumption that $t_0>L/v$. If $t_0<L/v$, then the discharge on the range $t_0 \leqq t \leqq L/v$ is given by

$$y=r\left\{\int_{x_0}^{tv} W(x)dx-e^{-\frac{t}{c}}\int_0^{tv} W(x)e^{\frac{x}{cv}}dx\right.$$
$$\left.+e^{-\frac{1}{c}\left(t-t_0\right)}\int_0^{x_0} W(x)e^{\frac{x}{cv}}dx\right\}. \qquad \text{(B–9)}$$

When $t=t_0$, equation (B–9) takes the form

$$y=r\left\{\int_0^{t_0v} W(x)dx-e^{-\frac{t_0}{c}}\int_0^{t_0v} W(x)e^{\frac{x}{cv}}dx\right\}; \qquad \text{(B–9a)}$$

and when $t=L/v$, equation (B–9) takes the form

$$y=r\left\{\int_{L-t_0v}^L W(x)dx-e^{-\frac{L}{cv}}\int_0^L W(x)e^{\frac{x}{cv}}dx\right.$$
$$\left.+e^{-\frac{1}{c}\left(L/v-t_0\right)}\int_0^{L-t_0v} W(x)e^{\frac{x}{cv}}dx.\right. \qquad \text{(B–9n)}$$

It should be noted that when $t=t_0$, equation (B–3) reduces to (B–9a); and when $t=L/v$, equation (B–6) reduces to (B–9b). Equations (B–9), (B–9a), and (B–9b) are the generalized forms of equations (9), (9a), and (9b), respectively.

It can be shown that the first derivative of equation (B–9) with respect to t can be written in the form

$$dy/dt=\frac{r}{c}e^{-t/c}\left\{\int_0^{tv} W(x)e^{\frac{x}{cv}}dx-e^{\frac{t_0}{c}}\int_0^{v(t-t_0)} W(x)e^{\frac{x}{cv}}dx\right\};$$

and the second derivative of equation (B–9) with respect to time can be written

$$\frac{d^2y}{dt^2}=-\frac{1}{c}\frac{dy}{dt}+\frac{rv}{c}\left\{W(tv)-W(tv-t_0v)\right\}.$$

If we set the first derivative just obtained equal to zero, we get the equation

$$e^{\frac{t_c}{c}}\int_0^{v(t_c-t_0)} W(x)e^{\frac{x}{cv}}dx=\int_0^{t_cv} W(x)e^{\frac{x}{cv}}dx. \qquad \text{(B–7a)}$$

By multiplying both sides of equation (B–9) by $e^{t/c}$ differentiating the resulting equation with respect to t, putting $dy/dt=0$, and lastly solving for y_c by multiplying by $\frac{1}{c}e^{-t-c}$, we get

$$y_c=r\int_{v(t_c-t_0)}^{t_cv} W(x)dx. \qquad \text{(B–8a)}$$

It follows from the conditions for a maximum that are given by the elementary calculus, and from the form of the second derivative of equation (B–9) above, that equation (B–8a) gives the maximum discharge when the solution of equation (B–7a) for t_c gives at least one value such that $t_0<t_c<L/v$, and when $W(t_cv)<W(t_cv-t_0v)$ for the value of t_c in question. If the width of the histogram of the drainage area increases with increasing distance from the gaging station, i. e., if $W(x)$ increases as x increases, even if in this case $W(x)$ is discontinuous, then $W(t_cv)>W(t_cv-t_0v)$ for all admissible values of t_c, and therefore equation (B–8a) cannot furnish the maximum discharge. If the width does not everywhere increase with increasing distance from the gage, the equation (B–8a) may furnish the maximum discharge.

A method of showing that the equations (B–3), (B–4), (B–6) and (B–5) are correct is to integrate them with respect to t between the limits for which they apply, and ascertain that the sum of the four integrals thus obtained is equal to the volume of rainfall which falls during the rain-causing freshet. This process was carried out in the first paper for the special case there discussed. In the same way it can be shown that the volume of discharge for the general case is $rt_0\int_0^L W(x)dx$, which is also the volume of rainfall.

Concerning equations (B–6) and (B–7) we can draw the following conclusions: If we evaluate $(dy/dt)_{t=t_0}$ for equation (B–6), we get $(r/c)e^{-t_0/c}\int_0^L W(x)e^{\frac{x}{cv}}dx$. Clearly this quantity is positive. If we evaluate $(dy/dt)_{t=t_0}+L/v$ for equation (B–6), we get $-(r/c)e^{-\frac{L}{cv}}(1-e^{-t_0/c})\int_0^L W(x)e^{x/cv}dx$. Clearly this quantity is negative. Now dy/dt is continuous for all values of t, even though $W(x)$ be discontinuous. Therefore, it follows that equation (B–7) has at least one real root t_c on the range $t_0\leqq t_c\leqq t_0+L/v$. Moreover equation (B–7) has only one real root on this range. For even if $W(x)$ be discontinuous, the left-hand member of equation (B–7) exists and is an increasing function of t_c. Since the left-hand member is an increasing function of t_c, it will equal the right-hand member at but one point. This root of equation (B–7) corresponds to a maximum and not a minimum. For the second derivative of equation (B–6) is

$$\frac{d^2y}{dt^2}=(r/c)\Big\{-\frac{1}{c}e^{-\frac{t}{c}}\int_0^L W(x)e^{\frac{x}{cv}}dx+$$

$$\frac{1}{c}e^{-\frac{1}{c}(t-t_0)}\int_0^{(vt-t_0)} W(x)e^{\frac{x}{cv}}dx-vW(tv-t_0v)\Big\}.$$

Since the first two terms vanish when $dy/dt=0$, and since $W(x)$ is not less than zero, it must follow that $d^2y/dt^2<0$ when $dy/dt=0$, and we have the necessary conditions for a maximum fulfilled. Thus the unique solution of equation (B–7) is the time of the crest. We have now proved that equation (B–6) has one and only one maximum point. By similar reasoning it can be shown that on the range $t_0\leqq t\leqq t_0+L/v$ where $t_0>L/v$, the discharge curve has no point of inflection at which the change in discharge with respect to time is zero.

We now turn to a similar discussion of equation (B–9) and its derivative. The first derivative of this equation evaluated at $t=t_0$ is $\frac{r}{c}e^{-\frac{t_0}{c}}\int_0^{t_0v} W(x)e^{\frac{x}{cv}}dx$. Clearly this quantity is positive. Moreover dy/dt is everywhere continuous. Now the first derivative evaluated at the point $t=L/v$ is

$$\frac{r}{c}e^{-\frac{L}{cv}}\Big\{\int_0^L W(x)e^{\frac{x}{cv}}dx-e^{\frac{t_0}{c}}\int_0^{L-t_0v} W(x)e^{\frac{x}{cv}}dx\Big\}.$$

Clearly this quantity is positive or negative according as the first term in the bracket is greater or less than the second term. If $(dy/dt)_{t=L/v}$ is negative, then dy/dt, being continuous, will have an odd number of real roots. If $(dy/dt)_{t=L/v}$ is positive, then dy/dt will have either no real root or an even number of real roots.

Consider now the range $L/v\leqq t\leqq t_0+L/v$ where $t_0<L/v$. On this range the discharge is given by an equation of the *same literal form* as equation (B–6); but as equation (B–6) was derived on the assumption that $t_0>L/v$, we shall, for the sake of definiteness, call the equation of discharge on this range where $t_0<L/v$ equation (B–6a). Recall here that when $t=L/v$, equation (B–6) and hence also equation (B–6a) reduce to equation (B–9b). Now if $(dy/dt)_{t=L/v}$ is negative then equation (2–6a) will not have a maximum. For its derivative is everywhere continuous, and as $e^{t_0/c}\int_0^{v(t-t_0)}W(x)e^{x/cv}dx$ is an increasing function of t, and all other quantities in the derivative of equation (B–6a) are positive, it follows that once this derivative becomes negative it remains negative. Furthermore, if $(dy/dt)_{t=L/v}$ is positive, then by the same reasoning as that used for equation (B–6) it follows that equation (B–6a) will have one and only one maximum point.

We now want two tests: (1) one to show whether $(dy/dt)_{t=L/v}\gtreqless 0$; and (2) the other, to be used when $(dy/dt)_{t=L/v}>0$, to show whether the first derivative of equation (B–9) will have no root or an even number of roots.

Test (1) is easily devised, thus: Solve equation (B–7) for t_c. If $t_cv\geqq L$, then $(dy/dt)_{t=L/v}\geqq 0_5$ and equation (B–6a) furnishes a maximum. The convenient thing about this test is that the test itself furnishes the time of the maximum discharge, provided of course $(dy/dt)_{t=L/v}\geqq 0$ and not <0. If $t_cv<L$, then equation (B–6a) will not have a maximum.

It appears impracticable to devise a convenient test for (2) to use on the general case of $W(x)$. However, a few special cases of $W(x)$ will be discussed:

Case I: Suppose that in addition to the conditions stated at the beginning of this section, $W(x)$ is such that at discontinuities $W(x-0)>W(x+0)$ and that $(d/dx)W(x)$ is never positive. In this case equation (B–9) may have a maximum, but it can never have more than one maximum. For since both equation (B–9) and its first derivative are continuous, then in order for it to have two maxima it must have one minimum. Now in this case d^2y/dt^2 satisfies the conditions for a maximum point everywhere, but at no point are the conditions for a minimum point satisfied.

Case II: Suppose that in addition to satisfying the conditions first stated, $W(x)$ also satisfies the following conditions: to the *left* of a certain point, say $x=a_1$, $W(x)$ is such that at discontinuities $W(x-0)<W(x+0)$, and $(d/dx)W(x)$ is never negative; while to the *right* of this point $W(x)$ is such that at discontinuities $W(x-0)>W(x+0)$, and $(d/dx)W(x)$ is never positive. Then in this case equation (B–9) may have a maximum, but it can never have more than one maximum. For it cannot have a maximum such that vt_c lies to the left of $x=a_1$; and to the right of a_1, the reasoning in the first case holds.

Case III: Suppose that in addition to the conditions first stated $W(x)$ contains a point, say $x=b$, to the left of which $W(x)$ satisfies the conditions of case II (the point a_1 may of course be the point $x=0$), and to the right of which $W(x)$ also satisfies the conditions of case II (the greatest value of $W(x)$ to the right of b being at the point $x=a_2$), and such that the point $(b-0)$ furnishes a minimum on the range $a_1<x<b$ while the point $(b+0)$ furnishes a minimum on the range $b<x<a_2$. In this case equation (B–9) may have two maxima. For suppose equation (B–7a) yields three roots for t_c; then call these three roots t_{c1}, t_{c2}, and t_{c3}. Now, if $a_1<t_{c1}v<a_2$ and $W(t_{c1}v)<W(t_{c1}v-t_0v)$, we have a maximum. If $b<t_{c2}v<L$ and $W(t_{c2})>W(t_{c3}v-t_0v)$, we haye a minimum. Lastly, if $a_2<t_{c3}v<L$ and $W(t_{c2}v)>W(t_{c2}v-t_0v)$, we again have a maximum. In this case, if $t_0v>a_2$ then equation (B–9) can furnish but one maximum point.

When the drainage area is of such shape that a steady rain may cause two or more crests it will always be necessary to apply both test (1) and test (2). For if $(dy/dt)_{t=L/v}>0$, we have to determine whether equation (B–7a) has no root or an even number of roots; while if test (1) shows the inequality sign reversed, we know equation (B–7a) has an odd number of roots but we have to determine how many roots it has.

The question naturally arises whether a watershed can be of such shape that a crest may occur before the rain stops. In other words, can equation (B–3) furnish a maximum? Now the first derivative of equation (B–3) can be written

$$(r/c)e^{-t/c}\int_0^{tv} W(x)e^{\frac{x}{cv}}dx.$$

Clearly this expression never vanishes, and therefore no drainage area can be of such shape that a crest will occur before the rain stops. It should be emphasized here that we are assuming the rate of rainfall to be constant.

Equation (B–8) has a physical interpretation. It shows that the maximum discharge from the entire drainage area is equal to the discharge from that portion of the drainage area lying between two equal water travel lines, corresponding to $x=L$ and $x=v(t_c-t_0)$, respectively, in a steady state. To explain this, put $t^*=t_c-t_0$. Now as $t_0\to\infty$, clearly t_c does also, because t_c is greater than t_0. However, as $t_0\to\infty$, then $t^*\to 0$; for if equation (B–7) be written in the form

$$\int_0^{t^*v} W(x)e^{\frac{x}{cv}}dx=e^{-t_0/c}\int_0^{L} W(x)e^{\frac{x}{cv}}dx,$$

then as $t_0\to\infty$ the right-hand side $\to 0$, and as this equation is true for all values of $t_0>0$ it therefore follows that the left-hand side $\to 0$ also; but in order for the left-hand side $t_0\to 0$, t^* must $\to 0$, as stated above. Moreover, t^* can have the value 0 for no other value of t_0.

Since $t^*\to 0$ as $t_0\to\infty$, it follows from equation (B–8) that $y_c\to r\int_0^L W(x)dx$ as $t_0\to\infty$. In other words, y_c approaches the same limit that y does in equation (B–4) as $t\to\infty$ in that equation. Thus the maximum discharge obtained from equation (B–8) cannot exceed the discharge at a steady state. Also we can see that as $t_0\to\infty$, the equal water travel line given by $x=t^*v$ approaches the gaging station as a limit.

The question arises as to how rapidly $y_c\to r\int_0^L W(x)dx$. Clearly from the above discussion, y_c approaches this limit as quickly as y in equation (B–4) approaches this same limit. The limit may be considered, practically, to have been reached when the second term inside the brackets of equation (B–4) is negligible compared to the first term. Now,

$$\int_0^L W(x)e^{\frac{x}{cv}}dx<\int_0^L W(x)e^{\frac{L}{cv}}dx,$$

whence y has approached the limit in question when $e^{-\frac{1}{c}(t+L/v)}$ is negligible compared to unity. Hence, if the rain lasts such a time that its duration, diminished by L/v, is 2.30______ times c, then the discharge has approached a value which is *more* than 0.9 of the steady state; if the duration less L/v is 4.60______ times c, the discharge has approached a value which is *more* than 0.99 of the steady state; if the duration less L/v is 6.90______ times c, the discharge has approached a value which is *more* than 0.999 of the steady state; and so on. The constant, 2.30______ is the natural logarithm of 10, i. e., the reciprocal of the modulus of common logarithms.

Having discussed the behavior of the maximum discharge for infinitely long rains, we now take up the maximum discharge for infinitely short rains, i. e., cloudbursts. If $t_0=0$, we write the fundamental equation thus: $cZ=AR-\int_0^t Zdt$; from this we develop the equation $Z=\frac{AR}{c}e^{-\frac{t}{c}}$. (Compare with the development of equation (2) in the first paper.) Then the discharge will be given by

$$y=(R/c)\int_0^{tv} W(x)e^{-\frac{1}{c}(t-x/v)}dx=(R/c)e^{-tc}\int_0^{tv} W(x)e^{\frac{x}{cv}}dx.$$

The first derivative of this equation is $-y/c+(Rv/c)W(tv)$, and the second derivative is $-(1/c)(dy/dt)+(Rv^2/c)W'(ty)$; hence if the equation is to furnish a proper maximum it is necessary that $W'(t_cv)$ be negative. Clearly $W'(x)$ may be positive for all values of x; in such a case the point of maximum discharge is a cusp on the discharge curve. This cusp will be at the point $t=L/v$, and in this case the discharge curve does not have a proper maximum. If $W'(x)$ be negative when dy/dt vanishes, then the time of the crest will be given by the solution of the following equation for t_c:

$$cvW(t_cv)=e^{-\frac{t_c}{c}}\int_0^{t_cv} W(x)e^{\frac{x}{cv}}dx.$$

The maximum discharge for infinitely short rains can be found by substituting R/t_0 for r in equation (B–8a) and then finding the value of the right-hand side as $t_0 \to 0$ by the rule for evaluating indeterminate forms. Thus

$$\underset{t_0 \to 0}{\text{Limit}}\, y_c = \underset{t_0 \to 0}{\text{Limit}}\, R\left[v\frac{dt_c}{dt_0}W(t_cv) - v\left(\frac{dt_c}{dt_0}-1\right)W(t_cv-t_0v)\right]$$
$$= RvW(t_cv).$$

In words, the maximum discharge due to an infinitely short rain is equal to the product of the depth of the rain times the velocity of the water in the stream times the width of the drainage area corresponding to the point $x = t_cv$. Clearly, the width of the drainage area which enters in this product will vary with the capacity of the soil, but we can readily conclude that the maximum discharge from cloudbursts cannot exceed the product of the rainfall times the velocity of the stream times the maximum width of the drainage area. If the drainage area be of such character that equation (B–8a) cannot furnish the maximum discharge, then for infinitely short rains

$$y_c = (R/c)e^{-\frac{L}{cv}}\int_0^L W(x)e^{\frac{x}{cv}}dx.$$

This section will be concluded with a few remarks about the continuity of the rate of discharge and the discharge tendency curves. If equations (B–3), (B–4), (B–5), (B–6) and (B–9) be differentiated with respect to t, then the rate of discharge curve is obtained. If we substitute the various limits of the ranges of these equations in their first derivatives, it will be noted that the rate of discharge curve is continuous at these points. By differentiating the rate of discharge curve with respect to time, we obtain the discharge tendency curve. It can be shown that the discharge tendency curve, when $t_0 > L/v$, is continuous at the point $t = L/v$ if $W(L)$ is zero; otherwise, it is continuous there. At the point $t = t_0$ it would be continuous were $W(O) = 0$, but this has been excluded from the physical conditions of the problem, hence the discharge tendency curve is discontinuous at $t—t_0$. At the point $t = t_0 + L/v$ it is continuous only if $W(L)$ is zero. It can also be shown that the above statements hold when $t_0 < L/v$. In addition to the above discontinuities, the discharge tendency curve will have other discontinuities if the drainage area curve be discontinuous. Thus that portion of the discharge tendency curve corresponding to equation (B–3) will have the same number of discontinuities as the drainage area curve. However, discontinuities in the drainage area curve will not cause any discontinuities in the rate of discharge curve; hence the rate of discharge curve is everywhere continuous.

SECTION 5: DRAINAGE AREA CURVES

We now turn to some practical considerations connected with the application of the theory in the last section. Suppose we have topographic maps of a drainage area and have carefully constructed a histogram from them and that it is then desired to fit a curve to this histogram. The question arises: What form of curve should be used? Or in other words, what function should be used for $W(x)$?

From the necessity of having equations (B–7) and (B–8) in tractable forms it is essential that both $W(x)$ and $W(x)e^{\frac{x}{cv}}$ be integrable in terms of functions for which tables now exist, preferably elementary functions. The elliptically shaped histogram discussed in section 3 is an illustration of a case in which $W(x)$ is integrable; but $W(x)e^{x/cv}$ is not integrable in terms of tabulated functions.

A number of functions suggest themselves for $W(x)$. We can use a trigonometric series (Harmonic Analysis), a polynomial, an exponential series, or a Gram-Charlier series. All of these satisfy the condition of tractable integrability, but each of them has the same defect which the case of the triangle brought out. In the case of the triangle (as well as these four series) equation (B–7) is transcendental. Of course one can solve a transcendental equation by numerical methods, and no doubt in some applications this will be the best way.

We can make equation (B–7) a polynomial by fitting a curve of the form

$$W(x) = (a_c + a_1x + a_2x^2 + a_3x^3 + \cdots + a_nx^n)e^{-\frac{x}{vc}}$$

to the histogram of the drainage area. As c is not a function of time during any one flood, but does vary from flood to flood, it is necessary to fit a different curve to the histogram for each flood. At first this procedure appears to be very tedious; but consider the following suggestions:

Suppose we fit the curve

$$W(x) = (a_c + a_1x + a_2x^2 + a_3x^3 + \cdots + a_nx^n)e^{-\frac{x}{c_0v_0}}$$

to the histogram, where c_0v_0 is the arithmetic mean value of all the cv that are ever to be expected in the river basin in question. This will be done once, and will apply to all future floods. In this way the constants c_0v_0 are assumed, so to speak, and then the a_i are determined from the histogram. Now when a particular flood occurs, we want to fit the curve

$$(a_0' + a_1'x + a_2'x^2 + a_3'x^3 + \cdots + a_n'x^n)e^{-\frac{x}{cv}}$$

to the histogram, where cv is now known from the conditions existing at the beginning of the rain. Since these two curves are to fit the same histogram we require that they be equal; equate, and multiply both sides of the equation thus obtained by $e^{x/cv}$:

$$(a_0 + a_1x + a_2x^2 + a_3x^3 + \cdots + a_nx^n)e^{-(x/c_0v_0 - x/cv)} = a_0' + a_1'x + a_{2'}x^2 + a_3'x^3 + \cdots + a_n'x^n$$

Now when $x/c_0v_0 - x/cv$ is small, the power series which represents the function $e^{-(x/c_0v_0 - x/cv)}$ is rapidly convergent; we assume this condition is fulfilled. Expand the exponential in the last equation in a power series, multiply together the two series on the left, and equate coefficients of like powers of x. On doing this we get the following system of equations:

$$\left.\begin{aligned}
a_0' &= a_0,\\
a_1' &= a_1 - (1/c_0v_0 - 1/cv)a_0,\\
a_2' &= a_2 - (1/c_0v_0 - 1/cv)a_1 + (1/c_0v_0 - 1/cv)^2\frac{a_0}{2},\\
a_3' &= a_3 - (1/c_0v_0 - 1/cv)a_2 + (1/c_0v_0 - 1/cv)^2\frac{a_1}{2}\\
&\quad - (1/c_0v_0 - 1/cv)^3\frac{a_0}{3},\\
a_4' &= a_4 - (1/c_0v_0 - 1/cv)a_3 + (1/c_0v_0 - 1/cv)^2\frac{a_2}{2}\\
&\quad - (1/c_0v_0 - 1/cv)^3\frac{a_1}{3} + (1/c_0v_0 - 1/cv)^4\\
&\quad \frac{a_0}{4}, \text{ etc.}
\end{aligned}\right\}\ \mathrm{I}$$

If $|x/c_0v_0-x/cv|$ is not sufficiently small to insure rapid convergence when the exponential is expanded in a power series, then, instead of fitting just one curve to the histogram, we can fit two or more curves to it. Thus, if two curves are fitted to the histogram, we divide the expected values of cv into two groups, one consisting of large values, the other small, and take the arithmetic mean value of each group. One mean is used in fitting one curve, and the other for the second curve. Then in a particular flood we use the curve for which $1/c_0v_0$ is the nearer to $1/cv$ in order to compute the a_i'.

After having obtained the a_i' from the a_i by means of equations (I), we next solve equation (B–7) for $v(t_c-t_0)$; this is now very quickly accomplished, because we can integrate the functions easily, and after the integration has been performed we can then readily solve the polynomial obtained. In following this method, equation (B–8) takes the form:

$$y_c=r\int_{t_cv-t_0v}^{L}\sum_{i=0}^{i=n}a_i'x^ie^{-\frac{x}{cv}}dx.$$

This expression is readily integrated in terms of elementary functions by the following substitution: Put $x/cv=\theta$, $dx=cvd\theta$; when $x=L$, $\theta=L/cv$; and when $x=v(t_c-t_0)$, $\theta=(1/c)(t_c-t_0)$; and we have

$$y_c=r\sum_{i=0}^{i=n}a_i'(cv)^{i+1}\int_{(1/c)(t_c-t_0)}^{L/cv}\theta^ie^{-\theta}d\theta.$$

If the histogram of the drainage area should have a discontinuity, it will be necessary to fit two different curves, each of the form

$$(a_0+a_1x+a_2x^2+a_3x^3+\text{------}+a_nx^n)e^{-x/cv},$$

to the histogram, one to the left of the discontinuity and one to the right of it.

SECTION 6: ADDITIONAL DISCHARGE EQUATIONS

Thus far all of the discharge equations have been derived on the assumptions that while it is raining equation (1), viz:

$$Z=Ar(1-e^{-t/c}),$$

gives the volume of rate of run-off; and after the rain stops equation (2), viz.:

$$Z=Z_0e^{-\frac{1}{c}(t-t_0)},$$

gives the volume of rate of run-off. Clearly, when evaporation is considered, or the rate of rainfall is not constant, these equations are no longer true. Furthermore, these equations have been derived on the fundamental assumption that the rate of run-off at any given time is proportional to the rainfall remaining with the soil at that time. While it is believed that this assumption is a very close approximation to what occurs in Nature, it is evident that different underlying assumptions will lead to different forms of equations (1) and (2), even where evaporation is neglected and the rate of rainfall considered constant. A few remarks will now be made about the discharge equations which result when the run-off equations are made more general.

Suppose while the rain is falling the volume of rate of run-off is given by the function $Z=AZ_1(t)$, and after the rain stops the volume of rate of run-off is given by $Z=AZ_2(t)$. For the present no restrictions will be placed on the functions $Z_1(t)$ and $Z_2(t)$ except that they be single-valued, or at points of discontinuity two-valued. Functions which are not single-valued would have no physical meaning.

In the next three papers special forms, nevertheless forms of greater generality than equations (1) and (2), of the functions $Z_1(t)$ and $Z_2(t)$ will be discussed. A portion of each of the next three papers will be taken to show that in most cases the practical problem of predicting flood crests does not warrant the use of run-off equations which are more general than equations (1) and (2). In this section, however, we consider the discharge from a watershed when the general run-off equations are used.

If while it is raining $Z_1(t)$ gives the rate of run-off, and if v be the velocity of the flowing water, a constant, then the discharge from a watershed whose drainage area curve is $W(x)$ is given by the equation

$$y=\int_0^{tv}W(x)\Sigma_1(t-x/v)dx,$$

which holds on the range $0\leqq t\leqq L/v$ and where $t_0\leqq L/v$. This equation is a generalization of equation (B–3).

Similarly, after the rain stops, if $Z_2(t)$ gives the rate of run-off, and v the velocity of the water, the discharge from the watershed is given by the equation

$$y=\int_0^{L}W(x)\Sigma_2(t-x/v)dx,$$

which holds on the range $t_0+L/v\leqq t\leqq\infty$. This is a generalization of equation (B–5).

Analogous equations can be written for the remaining ranges of t. The point to be emphasized is that the method of Section 4 is perfectly general, and involves the underlying relation between discharge and the rate of run-off. The rate of run-off is measured in inches per hour, and the discharge is measured in mile-inches per hour. The last two equations are merely symbolic forms of the following statement in words: The discharge from a drainage area at time t is the sum of the volumes of rate of run-off from all the infinitesimal strips above the gaging station, not at time t but at time t diminished by the interval required for the water to flow from where the rain falls to the gaging station; furthermore, the volume of rate of run-off from an infinitesimal strip is the product of the rate of run-off, be it any single-valued function of time whatever, times the length of the strip (width of drainage area), $W(x)$, times the width of the strip (the differential dx).

In the sixth paper of this series a variable velocity of the water will be considered. It will there be shown that for *special* variations of velocity all the conclusions of this paper, insofar as they relate to the prediction of flood crests, remain unchanged. The theoretical treatment of irregular-shaped drainage areas is now considered complete.

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